Fair and Efficient Allocation with Comparison Queries

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Abstract

Given a set of agents N, a set of indivisible items M, and a valuation function $v_i : 2^M \to \mathbb{R}$ for each agent $i \in N$, our goal is to compute a partition $(A_i)_{i \in N}$ of M that satisfies certain fairness and efficiency properties. Unlike in most work on discrete allocation, the valuation functions are accessed solely through *comparison queries*: an agent $i \in N$ is presented with two bundles $X, Y \subseteq M$ and reports whether $v_i(X) \leq v_i(Y)$. This query model captures the practical cost of preference elicitation while sparing agents the difficulty of assigning consistent numerical values to many bundles.

For additive valuation functions, we characterize the query complexity of finding an allocation with a given property in terms of its *certificate size* E, the minimum number of comparisons needed in the worst case to certify a valid allocation or the nonexistence of one. On convex domains of valuations, every zero-error algorithm requires at least E comparisons, whereas some deterministic algorithm uses only $O\left(E'|N||M|^2/\log E'\right)$, where $E' \coloneqq \max\{E, |M|+1\}$. Consequently, the existence of a valid allocation and a polynomial-query verifier together guarantee polynomial-query search.

Building on primitives from discrete geometry and active learning, we also develop polynomial-query algorithms for allocation problems that are NP-hard or co-NP-hard even with full information, such as searching for a *maximin share (MMS)* allocation and verifying *Pareto optimality (PO)*. Combining these primitives with the above characterization and known existence results, we further show that an allocation satisfying both *envy-freeness up to one item (EF1)* and PO can be found using $O\left(|N|^2|M|^4\right)$ comparisons when the additive valuation functions are either all monotone or all antitone.

We complement these results with query-complexity lower bounds that establish fundamental separations between allocation properties, valuation classes, query models, and numbers of agents. Notably, deciding whether three or more agents with monotone additive valuations admit an *envy-free (EF)* allocation requires $2^{\Omega(|M|)}$ comparisons even for randomized algorithms with bounded error. This contrasts not only with the value-query model, where $|N||M|$ queries suffice, but also with the polynomial comparison-query complexity of EF with two agents and of EF1 + PO with any number of agents.

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Table 1: Selected bounds on the number of comparison queries needed to find an allocation with a given property, or to report that none exists. Tables 2 to 4 (Appendix C) list our bounds for search, decision, and verification, together with existing ones. The number of items is denoted by m, and Section 3 defines the properties; a subscript c allows up to c unallocated items, and a superscript $+$ also counts items of zero marginal value. Abbreviations: Hom. = homogeneous; Mon. = monotone; Doubly Mon. = doubly monotone; Add. = additive; Submod. = submodular.

Property	Agents	Valuation	Upper Bound	Lower Bound
EF	2	Add.	$O\left(m^2 \log m\right)$ [Thm. 8]	$\Omega\left(m^2\right)$ [Thm. 12]
	n	Add.	$n\, 2^{O(m)}$ [Cor. 4]	$2^{\Omega(m)}$, $n \geq 3$ [Thm. 14]
	n (Hom.)	Add.	$O\left(m^{2.5} \log^2 m\right)$ [Thm. 8]	$\Omega\left(n + (m-n)^2\right)$, $m \geq n$ [Thm. 13]
		Mon. Submod.	$O\left(m^2 + 2.62^{m-n}\right)$, $m < 2n$ [Prop. 5]	$\Omega\left(n^2 + 2^{m-n}/m\right)$, $m > n$ [Thm. 34]
EFX_0	n	Mon. Add.	–	$\Omega\left(n\,(m-n)\right)$ [Thm. 18]
EFX_0^+	2	Add.	$O\left(m \log m\right)$ [Thm. 16]	$\Omega\left(m\right)$ [Thm. 19]
EFX_{n-3}^+	3, 4	Mon. Add.	$O\left(m^3/\log m\right)$ [Thm. 16]	–
EFX_{n-2}^+	n	Mon. Add.	$O\left(n^2 m^3/\log\left(nm\right)\right)$ [Thm. 16]	–
EF1	2	Any	$O\left(\log m\right)$ [Thm. 15]	$\Omega\left(\log m\right)$ (det.) [127, 39]
	n	Mon. Add.	$O\left(nm \log\left(m/n\right)\right)$, $m \geq 2n$ [107]	$\Omega\left(n \log\left(m/n\right)\right)$ (det.) [127] $\Omega\left(n \min\left\{n, m-n\right\}\right)$ [Thm. 17]
EFL	n	Doubly Mon./Add. Mon.	$O\left(nm \log \min\left\{n, m\right\}\right)$ [Thm. 15]	$\Omega\left(n \min\left\{n, m-n\right\}\right)$ [Thm. 17]
1-out-of-n MMS	2	Add.	$O\left(m^2 \log m\right)$ [Thm. 21]	$\Omega\left(m^2\right)$ [Cor. 5]
	n (Hom.)	Add.	$\tilde{O}\left(m^2 \min\left\{n, m\right\}\right)$ [Thm. 21]	$\Omega\left((m-n)^2\right)$, $m > n$ [Cor. 5]
1-out-of-d MMS	n	Add.	$O\left(nm^3 \log m\right)$ [Thm. 21]	–
RMMS	n	Add.	$O\left(nm^3 \log m\right)$ [Thm. 23]	$\Omega\left((m-n)^2\right)$, $m > n$ [Cor. 5]
PO	n	Mon. Add.	$O\left(n + m \log n\right)$ [Prop. 4]	$\Omega\left(n + m \log n\right)$ [Thm. 25]
		Mon. Submod.	$O\left(n\binom{m}{\lfloor m/2 \rfloor}\right)$ [Prop. 5]	$\Omega\left(n + \binom{m}{\lfloor m/2 \rfloor}\right)$ [Thm. 35]
XEF1/IEF1 + PO	n	Add.	$O\left(n \min\left\{n, m\right\} m^4\right)$ [Thm. 29]	$\Omega\left(nm\right)$ [Thm. 30]
EF1 + PO	n	Mon. Add.	$O\left(n \min\left\{n, m\right\} m^4\right)$ [Thm. 29]	$\Omega\left(n + m \min\left\{n, m\right\}\right)$ [Thm. 30]
EFX_0 + PO + MMS	2	Add.	$O\left(m^2 \log m\right)$ [Thm. 31]	$\Omega\left(m^2\right)$ [Cor. 5]
EFX_0^+ + PO	n (Hom.)	Add.	$O\left(m \log m\right)$ [Thm. 32]	$\Omega\left(m\right)$, $m > n$ [Thm. 19]
$\text{EFL}/\text{EFX}_m^+ + \text{RMMS}_m$	n	Mon. Add.	$O\left(n^2 m^4\right)$ [Thm. 33]	$\Omega\left((m-n)^2\right)$, $m > n$ [Cor. 5]
Any verifiable	n	Add., $\leq V$ values	$O\left(n\,(m+V) \log\left(V+1\right)\right)$ [Cor. 3]	–

1 Introduction

From property settlement to task assignment, the fair and efficient division of indivisible objects among heterogeneous agents is of ubiquitous importance [34, 124]. A wealth of work in algorithmic game theory has formulated this problem through allocation criteria defined by the agents' valuation functions, to which unlimited access is typically assumed [9, 82, 112]. In particular, the community's attention has been renewed by the celebrated results revealing the existence of an allocation that simultaneously satisfies compelling fairness and efficiency properties, namely *envy-freeness up to one item (EF1)* and *Pareto optimality (PO)*, when the agents' valuation functions are all monotone additive [42] or all antitone additive [120].

In many practical settings, however, eliciting the valuation functions can be a greater bottleneck than computing an allocation given full information. Assigning consistent numerical values to many bundles is cognitively demanding and often unreliable, whereas comparing two bundles requires only an ordinal judgment, without the need to calibrate utility magnitudes [27, 144, 98]. Pairwise comparisons have been studied extensively in preference elicitation and recommendation [53, 100],

learning valuation and utility functions [21, 80], active learning and ranking [97, 101, 91], and combinatorial optimization [51, 70], with more recent applications to language-model ranking and preference alignment [134, 45]. We therefore consider allocation under the *comparison-query* model: an algorithm may ask an agent only whether she weakly prefers one bundle to another, and its cost is the total number of such queries.

Comparisons are much weaker than *value queries* [31], with which an agent reports the value of a bundle: any comparison query can be simulated using two value queries, so lower (resp. upper) bounds on value- (resp. comparison-) query complexity imply those on the other. Of particular note is that a linear number of value queries can reconstruct an unknown additive function, whereas exponentially many comparisons are inevitable in the worst case to sort all subset sums [58]. Furthermore, comparison-query complexity does not follow computational complexity: while polynomially many comparisons may suffice for some NP-hard problems, there exist allocation properties (e.g., *proportionality*; Remark 5) that cannot be verified under the comparison model.

Although both query models have been addressed in the context of discrete fair division [129, 127, 39, 107, 158, 109, 108, 67], previous work revolves around property- and algorithm-specific analyses of a handful of fairness notions, mostly for two or three agents or for agents with identical valuations, despite the variety of solution concepts and instance classes addressed in the full-information setting. Meanwhile, a majority of existing approaches for achieving fair and efficient allocations [26, 42, 23, 72, 17, 76, 60, 18, 74, 19, 77, 36, 75, 78, 119, 24, 120] rely heavily on cardinal information or non-constructive arguments and do not readily yield polynomial-query algorithms in the comparison model. These limitations give rise to the following questions underlying our work:

> Can we compute an allocation that simultaneously satisfies fairness and efficiency using polynomially many comparisons? More generally, what characterizes the query complexity of an allocation problem, and when is it polynomial?

1.1 Our Contributions

Certificates Govern Comparison Complexity. The answers that a correct algorithm receives prove that its output is valid, or that no valid allocation exists, so the *certificate size*, the worst-case minimum number of comparisons whose answers certify a correct outcome, bounds the number of queries from below. We show that this bound is nearly tight: for every property determined by comparisons, on any product of convex sets of additive valuations, possibly with some ties excluded, finding a valid allocation, or reporting that none exists, takes between E and $O\left(Enm^2/\log E\right)$ comparisons for the certificate size $E \geq m+1$, with the lower bound even for zero-error randomized algorithms (Theorem 1). The polynomial overhead cannot be dropped: some predicate with certificates of size 3 needs $\Theta\left(nm\right)$ comparisons, even with bounded error (Remark 1). The upper bound builds on a majority-vote argument of Moshkov [122], which Hegedűs [86] combined with Helly's theorem to learn halfspaces using membership queries; its short refutations are also related to the polynomial certificates of Hellerstein et al. [87]. We extend this approach to search problems with potentially multiple valid outputs, run it with comparison queries alone over products of convex domains, and prove the lower bound for randomized algorithms. Because a correct verifier certifies the allocations that it accepts, existence and polynomial-query verification together give polynomial-query search (Corollary 1); unlike the optimization from certification of Cohen-Addad et al. [51], our search algorithm never runs the verifier. Certificates are also geometric: the order of all bundles at a generic valuation is certified exactly by the comparisons corresponding to the facets of its chamber in a hyperplane arrangement, as for threshold functions [162, 11, 10], so sorting all bundles takes between $f(m)$ and $O\left(mf(m)\right)$ comparisons, where the largest number $f(m)$ of facets of a chamber grows with a base between $\varphi \approx 1.618$ and $\vartheta \approx 1.709$ (Corollary 2).

Exponential Hardness of Envy-Freeness. Our main lower bound (Theorem 14) uses this geometry: with three or more agents with monotone additive valuations and $m \geq 5$ items, deciding whether an EF allocation exists requires at least $F_m = 2^{\Omega(m)}$ comparisons, even for zero-error randomized algorithms, and $(1 - 2\varepsilon) F_m$ with error probability ε, as the construction assigns values to a core of items according to a point in a chamber with F_m facets [47], each of which a certificate of nonexistence must rule out. In contrast, nm value queries reveal the whole instance, EF allocations have certificates of $n(n-1)$ comparisons, and polynomially many comparisons find EF allocations for two agents and for identical valuations (Theorem 8). For every fixed $n \geq 3$, the comparison-query complexity of EF on monotone non-degenerate additive instances grows at the exponential rate of $f(m)$ (Remark 7).

Algorithms and Primitives. Our upper bounds rest on primitives for subset sums: determining how every bundle compares with a fixed bundle uses only $O(m^2 \log m)$ comparisons, by point location in a hyperplane arrangement [103] (Theorem 2), which yields optimizers and threshold selectors, e.g., for maximin shares (Section 2.5). With them, verifying PO takes $O(nm^2 \log m)$ comparisons (Theorem 24), finding an exact 1-out-of-d MMS allocation, or reporting that none exists, $O(nm^3 \log m)$ (Theorem 21), and finding an EF allocation $O(m^2 \log m)$ for two agents and $O(m^{2.5} \log^2 m)$ for identical valuations (Theorem 8), although these problems are NP-hard or co-NP-hard with full information. With the existence results of Barman and Verma [24], Corollary 1 gives polynomial-query algorithms for XEF1 + PO and IEF1 + PO on all additive instances (Theorem 29); as these notions coincide with EF1 on goods and on chores, respectively, this finds EF1 + PO allocations [42, 120] with polynomially many comparisons, although no polynomial-time algorithm is known. For two agents with any additive valuations, an EFX + PO allocation that also gives both agents their maximin shares takes $O(m^2 \log m)$ comparisons (Theorem 31).

Separations. Our lower bounds separate allocation properties, valuation classes, query models, and numbers of agents. In addition to Theorem 14, finding an EF allocation for two agents takes $\Omega(m^2)$ comparisons even with bounded error (Theorem 12), whereas $2m$ value queries suffice; on goods, finding a PO allocation takes $\Theta(n + m \log n)$ comparisons (Proposition 4 and Theorem 25), an EF1 allocation for two agents only $O(\log m)$ (Theorem 15), and an EF1 + PO allocation $\Omega(n + m \min\{n, m\})$ (Theorem 30). With monotone submodular valuations, deciding whether an EF allocation exists takes $2^{\Omega(m-n)}$ queries and finding or verifying a PO allocation $2^{\Omega(m)}$, even for randomized algorithms with value queries and, except for PO search, for identical valuations (Theorems 34 and 35 and Remark 12), and the PO bounds are tight for a fixed number of agents (Proposition 5). Table 1 summarizes selected bounds for search, and Table 2 in Appendix C lists our search bounds together with existing ones; Section 4 discusses open questions.

1.2 Related Work

Query models have a long history in the well-studied cake-cutting problem [132], most notably through the Robertson–Webb model [137, 155, 146, 61, 131, 57, 14, 35, 90]. For envy-free cake cutting, recent work has reduced the number of queries from a tower of exponentials [14] to a single exponential [159] and, with randomization, to a polynomial [116]. This line of work has also inspired query-efficient approaches for fair division of indivisible items via discrete adaptations of the cut-and-choose, moving knives, and binary search methods [34, 145, 155, 35].

For indivisible goods, Plaut and Roughgarden [129] prove that finding an EFX (envy-free up to any item) allocation may require exponentially many value queries for two agents even with identical valuations. Oh et al. [127] obtain EF1 allocations using a polylogarithmic number of value

queries for two agents with monotone valuations and for three agents with additive valuations, while proving lower bounds for EF, EF1, and EFX. Bu et al. [39] obtain comparison-query algorithms that use $O(\log m)$ queries for a constant number of agents, for PROP1 (proportionality up to one item) together with 1/2-MMS, and for EF1 in several cases, including identical additive valuations. Yamagata and Sumita [158] show that comparisons with arbitrary tie-breaking may not suffice even to verify EF1, in contrast to our comparison queries, which allow ties to be detected. They compute EF1 allocations together with explicit witness certificates for identical monotone valuations with $O(n \log n \log(m/n))$ comparison queries if $m > 2n$, with a 1/2-MMS guarantee as well for identical additive valuations. For additive chores, Lin et al. [109] compute PROP1 allocations with $O(n^3 \log m)$ comparison queries, $(13/11 + \varepsilon)$-MMS allocations for a fixed number of agents with a number of queries logarithmic in m, and EF1 allocations for three agents with $O(\log m)$ queries. Li et al. [107] analyze the round-robin algorithm under both the value-query and the item-comparison model and obtain almost matching query complexity bounds, along with extensions to noisy settings; their subsequent work [108] studies value queries with stochastic noise for envy-free allocation with two agents. Filos-Ratsikas et al. [67] find α-EFX allocations for additive valuations from the agents' ordinal rankings of the items and a few value queries, with near-optimal trade-offs between α and the number of queries.

Liu et al. [113] compute EF1, PROP1, and EFX allocations in a different interactive setup in which the algorithm repeatedly proposes an allocation and an agent reports a violation witness unless the allocation satisfies the property. The two models distribute the burden differently between agents and the algorithm: in their model, agents must identify violations, which can be demanding for properties such as PO and MMS, whereas our comparison queries require only pairwise preference judgments, leaving the algorithm responsible for certifying the property.

In the communication model, where each agent knows her valuation, Plaut and Roughgarden [130] determine whether computing a fair allocation, or deciding that none exists, requires polynomial or exponential communication, for envy-freeness, proportionality, and their approximations with submodular, subadditive, and general valuations. Feige [64] gives randomized protocols with an expected average of $O(\log m)$ bits per agent for approximately proportional allocations with additive valuations, and lower bounds such as $\Omega(m/n)$ bits per agent on average for the minimum EFX share. Since an algorithm making q comparison queries gives a protocol in which the agents broadcast the q answer bits, with the same randomness and error, our upper bounds carry over to communication, while communication lower bounds also apply to comparison-query complexity.

Appendix C discusses further related work: allocation of indivisible items with full information (C.1); preference elicitation and valuation learning (C.2); ranking, sorting, and selection from comparisons (C.3); and comparison oracles in active learning, geometry, and optimization (C.4).

1.3 Model and Notation

For a finite set X and an integer k, we write $\binom{X}{k}$, $\binom{X}{\leq k}$, and $\binom{X}{\geq k}$ for the families of subsets of X with exactly, at most, and at least k elements, respectively.

Definition 1. For a finite set M, a function $v : 2^M \to \mathbb{R}$ is said to be *queryable* if it accepts *comparison queries*, each of which takes subsets $X, Y \subseteq M$ and returns whether $v(X) \leq v(Y)$. The function v is called *monotone* (resp. *antitone*) if $v(S) \leq v(T)$ (resp. $v(S) \geq v(T)$) for any $S \subseteq T \subseteq M$, and *additive* if $v(S) = \sum_{g \in S} v(\{g\})$ for any $S \subseteq M$, when we equate it with the vector $(v(\{g\}))_{g \in M} \in \mathbb{R}^M$. For each subset $S \subseteq M$, we define $\mathbf{1}_{M|S} \in \{0,1\}^M$ so that $\mathbf{1}_{M|S}(g) = 1$ if and only if $g \in S$. For each $d \in \{0, 1, \ldots\}$, the set of all *d-partitions* of M is $\Pi_d(M) := \left\{ (B_i)_{i=1}^d \in \left(2^M\right)^d \mid |\{i \in \{1, 2, \ldots, d\} \mid g \in B_i\}| = 1 \text{ for every } g \in M \right\}$.

Definition 2. An *instance* $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$ consists of a finite set of *agents* $N = \{1, 2, \ldots, |N|\}$ with $|N| \geq 2$, a finite set of *items* M with $|M| \geq 2$, and a queryable *valuation function* $v_i : 2^M \to \mathbb{R}$ for each agent $i \in N$. The instance $\mathcal{I}$ is called *additive* (resp. *monotone*/*antitone*) if v_i is additive (resp. monotone/antitone) for each $i \in N$, *homogeneous* if $v_i = v_j$ for every $(i, j) \in N^2$, and *non-degenerate* if $v_i(X) \neq v_i(Y)$ for each $i \in N$ and $\{X, Y\} \in \binom{2^M}{2}$. For each $c \in \{0, 1, \ldots\}$, we define $\mathcal{A}_c(\mathcal{I}) \coloneqq \bigcup_{M' \in \binom{M}{\geq |M|-c}} \Pi_{|N|}(M')$. Each $A \in \mathcal{A}(\mathcal{I}) \coloneqq \mathcal{A}_{|M|}(\mathcal{I})$ is called an *allocation* in $\mathcal{I}$.

Definition 3. Let $\mathcal{C}$ be a set of instances. A *predicate* $\mathcal{P}$ on $\mathcal{C}$ is a function that maps each instance $\mathcal{I} \in \mathcal{C}$ to a subset $\mathcal{P}(\mathcal{I}) \subseteq \mathcal{A}(\mathcal{I})$. For any predicates $\mathcal{P}$ and $\mathcal{P}'$ on $\mathcal{C}$, their *intersection* $\mathcal{P} \cap \mathcal{P}'$ is the predicate on $\mathcal{C}$ that maps each instance $\mathcal{I} \in \mathcal{C}$ to $\mathcal{P}(\mathcal{I}) \cap \mathcal{P}'(\mathcal{I})$. We consider (possibly randomized) algorithms Λ that are given the set $\mathcal{C}$, the predicate $\mathcal{P}$, the sets of agents and items of an unknown instance $\mathcal{I} \in \mathcal{C}$, and, for verification, an allocation $A \in \mathcal{A}(\mathcal{I})$, and that access the valuation functions of $\mathcal{I}$ only through queries. The *verification complexity* of $\mathcal{P}$ by Λ, $\mathrm{VR}(\mathcal{P} \mid \Lambda)$, is the supremum, over all $\mathcal{I} \in \mathcal{C}$ and $A \in \mathcal{A}(\mathcal{I})$, of the expected[1] number of queries that Λ uses to verify whether $A \in \mathcal{P}(\mathcal{I})$; the *decision complexity* of $\mathcal{P}$ by Λ, $\mathrm{DC}(\mathcal{P} \mid \Lambda)$, is the supremum, over all $\mathcal{I} \in \mathcal{C}$, of the expected number of queries that Λ uses to decide whether $\mathcal{P}(\mathcal{I}) = \emptyset$; the *search complexity* of $\mathcal{P}$ by Λ, $\mathrm{SR}(\mathcal{P} \mid \Lambda)$, is the supremum, over all $\mathcal{I} \in \mathcal{C}$, of the expected number of queries that Λ uses to verify that $\mathcal{P}(\mathcal{I}) = \emptyset$ or compute any $A \in \mathcal{P}(\mathcal{I})$. For each $\mathrm{Task} \in \{\mathrm{VR}, \mathrm{DC}, \mathrm{SR}\}$, $\mathrm{Task}_{\mathrm{det}}(\mathcal{P})$ (resp. $\mathrm{Task}_{\mathrm{rand}}(\mathcal{P})$) denotes the infimum of $\mathrm{Task}(\mathcal{P} \mid \Lambda)$ over all such deterministic (resp. randomized) algorithms Λ that perform the task correctly with probability 1 for every $\mathcal{I} \in \mathcal{C}$. For $\varepsilon \in [0, 1/2)$, $\mathrm{Task}_\varepsilon(\mathcal{P})$ denotes the infimum of $\mathrm{Task}(\mathcal{P} \mid \Lambda)$ over all randomized algorithms Λ that perform the task correctly with probability at least $1 - \varepsilon$ for every $\mathcal{I} \in \mathcal{C}$ and, for verification, every $A \in \mathcal{A}(\mathcal{I})$; thus $\mathrm{Task}_\varepsilon(\mathcal{P}) \leq \mathrm{Task}_0(\mathcal{P}) = \mathrm{Task}_{\mathrm{rand}}(\mathcal{P}) \leq \mathrm{Task}_{\mathrm{det}}(\mathcal{P})$.

1.4 Organization of the Paper

In Section 2, we view comparison queries as signs in valuation space, show that certificate size governs the comparison complexity of every property determined by comparisons on convex domains, bound the cost of sorting all bundles, and develop the subroutines for point location, optimization, and selection that we use later, together with the principles behind our lower bounds. Section 3 leverages these tools to derive query complexity bounds for an array of selected allocation problems, organized by property, and complements them with lower bounds for the additive and submodular cases. Section 4 discusses runtime, the consequences of further existence results, and open questions. Some technical details are deferred to Appendices A and B, and Appendix C discusses further related work and compares our bounds with existing ones in Tables 2 to 4.

2 Comparisons in Valuation Space

This section develops the tools behind our bounds. For additive valuations, a comparison query reveals the sign of a linear form with coefficients in $\{-1, 0, 1\}$, so a comparison algorithm locates the unknown valuation functions in an arrangement of hyperplanes (Section 2.1). Section 2.2 shows that the comparison complexity of every property determined by comparisons on convex domains is governed by the size of its certificates, so that existence and polynomial-query verification give polynomial-query search. Section 2.3 bounds the exponential cost of sorting all bundles by the largest number of facets of a chamber, and Sections 2.4 and 2.5 provide polynomial-query subroutines for one-pivot point location, optimization, and selection behind our algorithms. Section 2.6 collects the principles behind our lower bounds, also for randomized algorithms.

[1] Here, expectations are over the internal randomness of Λ, if any.

2.1 Comparisons as Signs

Throughout this section, let N and M be finite sets with $n \coloneqq |N|$ and $m \coloneqq |M| \geq 2$, and equate each additive valuation function with its vector in $\mathbb{R}^M$. Let $\mathcal{H}_M \coloneqq \{-1, 0, 1\}^M \setminus \{\mathbf{0}\}$ denote the set of *ternary normals*. For an additive function v and subsets $X \neq Y$ of M, we have $v(X) - v(Y) = \langle v, h \rangle$ with $h \coloneqq \mathbf{1}_{M|X} - \mathbf{1}_{M|Y} \in \mathcal{H}_M$, and every $h \in \mathcal{H}_M$ arises in this way. Thus a comparison query to agent i about X and Y reveals the bit $[\langle v_i, h \rangle \leq 0]$, which depends only on the *coordinate* $(i, h) \in N \times \mathcal{H}_M$. We call the bit vector

$$\beta(\mathcal{I}) \coloneqq ([\langle v_i, h \rangle \leq 0])_{(i,h) \in N \times \mathcal{H}_M} \in \{0,1\}^{N \times \mathcal{H}_M} \tag{1}$$

the *type* of an additive instance $\mathcal{I}$ with agents N and items M. Instances of the same type answer all comparison queries alike, so a deterministic algorithm behaves identically on them. A predicate $\mathcal{P}$ on a set $\mathcal{C}$ of such instances is *determined by comparisons* if $\mathcal{P}(\mathcal{I}) = \mathcal{P}(\mathcal{I}')$ whenever $\beta(\mathcal{I}) = \beta(\mathcal{I}')$. Every predicate that some deterministic algorithm verifies with comparison queries is determined by comparisons, and so is every property considered in this paper.

Geometrically, the types of a single agent correspond to the faces of the central arrangement of the hyperplanes $\{x \in \mathbb{R}^M \mid \langle x, h \rangle = 0\}$ with $h \in \mathcal{H}_M$, which we call the *ternary arrangement*. Valuation vectors without ties, i.e., those of non-degenerate instances, lie in its full-dimensional faces, called *chambers*; each chamber is an open pointed polyhedral cone, and its *facets* are the faces of codimension one of its closure, each spanning a hyperplane of the arrangement. Let $f(m)$ denote the largest number of facets of a chamber. Since flipping the sign of a coordinate preserves the arrangement, it suffices to consider chambers in the positive orthant, which represent the additively representable comparative probability orders on m atoms studied by Fishburn et al. [68] and Christian et al. [48]. Chevyrev et al. [47, Theorem 3, Proposition 1, Corollaries 1 and 2, and Lemma 4] show that

$$F_{m+1} \leq f(m) \leq B_m \coloneqq \sum_{j=0}^{s_m} \binom{m}{j}, \quad \text{where } s_m \coloneqq \min\left\{ s \in \{0,1,\ldots\} \;\middle|\; \sum_{j=0}^{s} 2^j \binom{m}{j} \geq 2^m - 1 \right\} \tag{2}$$

and F_{m+1} is the $(m+1)$-st Fibonacci number ($F_1 = F_2 = 1$), so that $f(m)$ grows exponentially with a base between the golden ratio $\varphi \approx 1.618$ and $\vartheta \coloneqq 2^{H(\lambda)} \approx 1.709$, where $H(x) \coloneqq -x \log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function and $\lambda \in (0, 1/2)$ solves $\lambda + H(\lambda) = 1$; Searles conjectures $f(m) = F_{m+1}$ [47, Section 3]. Most chambers have far fewer facets than the largest ones: on average, a chamber has between m and $2m-2$ facets [73, Corollary 1.3]; see also Roudneff [138, Theorem 3.2] and Anthony [10, Theorem 4.14].

Two facts about the ternary arrangement drive our upper bounds, and their proofs are deferred to Appendix A.1. First, a vector of answers that no valuation realizes is refuted by few of its answers. For a single agent, the type of $x \in \mathbb{R}^M$ is $([\langle x, h \rangle \leq 0])_{h \in \mathcal{H}_M}$, and a vector $b \in \{0,1\}^{\mathcal{H}_M}$ *agrees* with x on $Q \subseteq \mathcal{H}_M$ if $b_h = [\langle x, h \rangle \leq 0]$ for every $h \in Q$.

Lemma 1. *Let $D \coloneqq K \setminus \bigcup_{h \in S} \{x \in \mathbb{R}^M \mid \langle x, h \rangle = 0\}$ for a convex set $K \subseteq \mathbb{R}^M$ and a subset $S \subseteq \mathcal{H}_M$. If $b \in \{0,1\}^{\mathcal{H}_M}$ is the type of no $x \in D$, then there exists $Q \in \binom{\mathcal{H}_M}{\leq m+1}$ such that b agrees with no $x \in D$ on Q.*

This is Helly's theorem [88] applied to the halfspaces that the answers prescribe; Hegedűs [86, Lemma 5.1] uses the same argument for halfspaces over a grid, and Moshkov [123, Lemma 6.17] proves the analogous statement for sign conditions on affine forms. Second, the number of types is singly exponential in nm^2, by a variant of Cover's function-counting theorem [54].

Lemma 2. *The ternary arrangement has at most* $1 + 2 \cdot 3^{m(m-1)} < 2^{1.6m^2}$ *faces. Therefore, any set of additive instances with agents* N *and items* M *has fewer than* $2^{1.6nm^2}$ *types, and fewer than* $2^{1.6m^2}$ *if its instances are homogeneous.*

2.2 Certificates and Query Complexity

The comparison answers that a correct algorithm receives prove that its output is valid. We measure how many answers such a certificate needs: for example, comparing $v_i(A_i)$ with $v_i(A_j)$ for all $i \neq j$ certifies that A is EF (Definition 5), whereas proving that no EF allocation exists can take F_m comparisons (Theorem 14).

Definition 4. Let $\mathcal{C}$ be a set of additive instances with agents N and items M, and let $\mathcal{P}$ be a predicate on $\mathcal{C}$ determined by comparisons. For $\mathcal{I} \in \mathcal{C}$ and $A \in \mathcal{P}(\mathcal{I})$, a set $Q \subseteq N \times \mathcal{H}_M$ of coordinates is a *certificate* of A at $\mathcal{I}$ if $A \in \mathcal{P}(\mathcal{I}')$ for every $\mathcal{I}' \in \mathcal{C}$ whose type agrees with $\beta(\mathcal{I})$ on Q. If $\mathcal{P}(\mathcal{I}) = \emptyset$, then Q is an *emptiness certificate* at $\mathcal{I}$ if $\mathcal{P}(\mathcal{I}') = \emptyset$ for every such $\mathcal{I}'$. The *certificate size* $E(\mathcal{P})$ is the maximum, over $\mathcal{I} \in \mathcal{C}$ with $\mathcal{P}(\mathcal{I}) \neq \emptyset$, of the least size of a certificate of some $A \in \mathcal{P}(\mathcal{I})$ at $\mathcal{I}$, and $E_\emptyset(\mathcal{P})$ is the maximum, over $\mathcal{I} \in \mathcal{C}$ with $\mathcal{P}(\mathcal{I}) = \emptyset$, of the least size of an emptiness certificate at $\mathcal{I}$; each maximum is 0 if there is no such $\mathcal{I}$.

Clearly, $\mathrm{SR}_{\mathrm{det}}(\mathcal{P}) \geq \max\{E(\mathcal{P}), E_\emptyset(\mathcal{P})\}$, since the queries of a correct deterministic algorithm certify its output. The following theorem shows that, on convex domains, the certificate size also bounds the search complexity from above, within a factor of $O(nm^2)$ of the certificate size or of $m+1$, whichever is larger, and that the lower bound holds for randomized algorithms; its proof is deferred to Appendix A.1. The domains it covers include all additive instances, monotone and antitone instances, strictly positive instances, and their non-degenerate subclasses, obtained by removing all the hyperplanes of the ternary arrangement.

Theorem 1. *For each* $i \in N$*, let* $D_i \coloneqq K_i \setminus \bigcup_{h \in S_i} \{x \in \mathbb{R}^M \mid \langle x, h \rangle = 0\}$ *be nonempty for a convex set* $K_i \subseteq \mathbb{R}^M$ *and a subset* $S_i \subseteq \mathcal{H}_M$*. Let* $\mathcal{C}$ *consist of the additive instances with agents* N *and items* M *such that* $v_i \in D_i$ *for every* $i \in N$*, and let* $\mathcal{P}$ *be a predicate on* $\mathcal{C}$ *determined by comparisons. Let* $E \coloneqq \max\{E(\mathcal{P}), E_\emptyset(\mathcal{P})\}$ *and* $E' \coloneqq \max\{E, m+1, 4\}$*. Then*

$$E \leq \mathrm{SR}_{\mathrm{rand}}(\mathcal{P}) \leq \mathrm{SR}_{\mathrm{det}}(\mathcal{P}) \leq E' + \frac{1.6\,E'nm^2}{\log_2 E'} = O\left(\frac{E'nm^2}{\log E'}\right). \tag{3}$$

The same holds with m^2 *in place of* nm^2 *if* $\mathcal{C}$ *instead consists of the homogeneous additive instances with agents* N *and items* M *such that* $v_i \in D_1$ *for every* $i \in N$*.*

The same proof gives the analogous bounds for decision and verification, with certificates of the answers in place of certificates of allocations. The upper bound is a majority-vote argument of Moshkov [122], with proofs in Moshkov [123, Theorems 3.5 and 3.6]: to determine the class of a hidden column of a table by querying its entries, the optimal depth d satisfies $\mu \leq d \leq \mu + \mu \log_2 \nu / \log_2 \mu$ for $\mu \geq 4$, where ν is the number of columns and μ is the largest number of entries needed to certify or refute the class of a hypothetical column; Hegedűs [86, Section 3] and Bshouty and Makhoul [38, Lemmas 4 and 8] state it in learning-theoretic form. Moshkov combined it with short refutations and a count of sign patterns to build decision trees with arbitrary linear tests for problems decided by the signs of linear forms [122, Theorem 3], [123, Theorem 6.1], and Hegedűs [86, Lemma 5.1 and Corollaries 5.4 and 5.5] likewise used Helly's theorem and Cover's count to learn halfspaces with membership queries; these short refutations are the polynomial certificates of Hellerstein et al. [87, Section 3.1]. Theorem 1 adds the form for search, where several

outputs may be valid, a straightforward extension; the use of the comparison queries themselves, not arbitrary linear tests, over products of convex domains with or without ties; and the lower bound for randomized algorithms.

Since the queries of a correct verifier certify the allocations it accepts, also for randomized verifiers by the argument behind Theorem 1, existence and verification together yield search.

Corollary 1. *In the setting of Theorem 1, suppose that* $\mathcal{P}(\mathcal{I}) \neq \emptyset$ *for every* $\mathcal{I} \in \mathcal{C}$*, and let* $V' \coloneqq \max\{\mathrm{VR}_{\mathrm{rand}}(\mathcal{P}), m+1, 4\}$*. Then* $\mathrm{SR}_{\mathrm{det}}(\mathcal{P}) = O\left(V'nm^2/\log V'\right)$*, and* $O\left(V'm^2/\log V'\right)$ *for the homogeneous instances.*

Cohen-Addad et al. [51, Theorem 1.3] is the conceptual precedent of this reduction: they optimize an unknown linear function with comparison queries, given an oracle that, for any explicitly given weight vector, returns an optimal solution together with a certificate of its optimality. Corollary 1 instead starts from an arbitrary verifier, which enters only through the certificate size that it bounds: the search algorithm is that of Lemma 6 and never runs the verifier. This algorithm is explicit if the types realized in each D_i can be listed together with an instance of each, e.g., by linear programming when each K_i is a polyhedron, and $\mathcal{P}$ can be evaluated on these instances; however, it maintains up to $2^{1.6nm^2}$ consistent types and finds each set of queries by exhaustive search, which takes exponential time.

Remark 1. The overhead factor cannot be replaced by a function of the certificate size alone. For all n and m such that $n\lfloor m/2 \rfloor$ is sufficiently large, Appendix A.2 constructs a predicate $\mathcal{P}$ on the additive instances with n agents and m items that satisfies $\mathcal{P}(\mathcal{I}) \neq \emptyset$ for every instance and has $E(\mathcal{P}) = \mathrm{VR}_{\mathrm{det}}(\mathcal{P}) = 3$, whereas $\mathrm{SR}_{\varepsilon}(\mathcal{P}) = \Theta(nm)$ for every fixed $\varepsilon \in [0, 1/2)$.

2.3 Sorting All Bundles

All the properties discussed in this paper are defined solely by (possibly an exponential number of) inequalities that comparison queries can verify. For the verification/decision/search problems of such properties, it suffices to sort all subsets of items according to each agent's valuation function. Learning the whole order of the bundles is the identification case of Theorem 1, as in the halfspace learner of Hegedűs [86, Corollaries 5.4 and 5.5]. Among valuations without ties, a set of comparisons determines the order at a valuation if and only if it contains a comparison across every facet of its chamber, since crossing a single facet changes only the comparisons across it; if alternatives with ties are allowed, one comparison per facet still suffices when it is asked in the orientation whose answer is a strict inequality. For threshold functions, the analogous statement that the facets form the unique minimal specifying set was observed by Cover [54] for points in general position and proved by Anthony et al. [11, Theorem 2.11 and Corollary 2.12]; Zuev [162, Section 5] noted it for the cones of threshold functions, and Anthony [10, Theorem 3.3] gives a recent account. For comparative probability orders, minimal sets of comparisons that determine an order consist of flippable pairs [68, Theorem 3.7], [48, Theorem 3], and, apart from at most one coordinate facet, the facets of the corresponding polytope correspond to the flips that are again representable [47, Proposition 1].

Corollary 2. *For a finite set* M *with* $m \geq 2$ *and a queryable additive function* $v : 2^M \to \mathbb{R}$*, sorting* 2^M *with respect to* v *requires* $f(m)$ *comparison queries in the worst case, even for randomized algorithms that are correct with probability* 1*, and* $(1-\varepsilon)f(m) - \varepsilon$ *for those that are correct with probability at least* $1-\varepsilon$*, where* $\varepsilon \in [0, 1/2)$*, whereas* $O(mf(m))$ *comparison queries suffice.*

Remark 2. For positive valuations without ties, Fishburn et al. [68] study the least number $c(m)$ of adaptive comparisons that always determines the order of 2^M, compute $c(1), \ldots, c(5) = 0, 1, 4, 9, 17$

[68, Theorem 4.2], and ask for its asymptotics [68, Problem 8]. Known results already give $c(m) = 2^{\Theta(m)}$: the upper bound $c(m) \le 2^{m+1} + 1.6m^2$ follows from Fredman [71, Theorem 3.1] and Lemma 2, and the lower bound $c(m) \ge 2^{\lfloor m/3 \rfloor}$ from Dietzfelbinger [58, Theorem 4.1], whose hard inputs are positive and generic. Corollary 2 narrows the base of the exponent from the interval between $2^{1/3}$ and 2 to that between φ and ϑ: its upper bound applies to $c(m)$, and its lower-bound argument gives $c(m) \ge F_{m+1}$ for $m \ge 4$ at the order of Chevyrev et al. [47, Theorem 3], which has F_{m+1} flippable pairs whose flips are all again additively representable, so that across each of the corresponding facets lies again a positive order.

Lemma 3 circumvents the exponential lower bound of Corollary 2 if the valuation functions take a polynomial number of distinct values, and Corollary 3 applies it to every property that comparisons can verify.

Lemma 3 ([51, Theorem 5.2 and its proof])**.** *Given a finite set M and a queryable additive function $v : 2^M \to \mathbb{R}$, one can sort 2^M with respect to v using $O\left((|M| + V)\log(V+1)\right)$ comparison queries, where $V \coloneqq |\{v(S) \mid S \subseteq M\}|$.*

Corollary 3. *Let $V \in \{1, 2, \ldots\}$, suppose that $\mathcal{C}$ consists of additive instances whose valuation functions each take at most V distinct values, and let $\mathcal{P}$ be any predicate on $\mathcal{C}$ that can be verified using comparison queries. Then $\max\{\mathrm{VR}_{\det}(\mathcal{P}), \mathrm{DC}_{\det}(\mathcal{P}), \mathrm{SR}_{\det}(\mathcal{P})\} = O\left(n\,(m+V)\log(V+1)\right)$.*

For example, if every item value is an integer in $\{-W, \ldots, W\}$ for some $W \in \{1, 2, \ldots\}$, then each valuation function takes at most $mW + 1$ distinct values, so Corollary 3 gives $O\left(nmW \log(mW)\right)$ queries; for fixed W, even $O(nm)$ queries identify the types of all agents (Appendix A.3). Corollary 3 also implies, when n is polynomial in m, that exponential lower bounds such as those of Corollary 2 and Dietzfelbinger [58] need exponentially many distinct bundle values.

2.4 One-Pivot Point Location

Sorting all bundles costs exponentially many comparisons (Section 2.3), but comparing every bundle with one fixed bundle, a *pivot*, takes only almost quadratically many, by point location in an arrangement of exponentially many hyperplanes, at a cost logarithmic in their number [103].

Theorem 2 provides a one-pivot subroutine for the selection tools of Section 2.5 and for verifying Pareto optimality (PO) and maximin shares (MMS) in Section 3: run with each agent's bundle as the pivot, it answers every comparison needed to verify them.

Theorem 2. *Given a finite set M with $|M| \ge 2$, a subset $S \subseteq M$, and a queryable additive function $v : 2^M \to \mathbb{R}$, one can compute $\left(\operatorname{sign}\left(v(T) - v(S)\right)\right)_{T \subseteq M} \in \{-1, 0, 1\}^{2^M}$ using $O\left(|M|^2 \log |M|\right)$ comparison queries.*

Remark 3. Up to a logarithmic factor, Theorem 2 matches an information-theoretic lower bound obtained by the following correspondence with boolean threshold functions. For $d \in \{1, 2, \ldots\}$, let

$$\mathcal{T} \coloneqq \left\{ \{0,1\}^d \ni x \mapsto \operatorname{sign}\left(\langle w, x\rangle - \theta\right) \in \{-1, 0, 1\} \mid (w, \theta) \in \mathbb{R}^{d+1} \right\} \tag{4}$$

denote the set of all boolean linear threshold functions in $\mathbb{R}^d$. For any $(w, \theta) \in \mathbb{R}^{d+1}$, the additive function v on $M \coloneqq \{1, 2, \ldots, d+1\}$ with $v(\{j\}) = w_j$ for $j \le d$ and $v(\{d+1\}) = \theta$ satisfies

$$\operatorname{sign}\left(\left\langle w, \mathbf{1}_{\{1,2,\ldots,d\}|T}\right\rangle - \theta\right) = \operatorname{sign}\left(v(T) - v(\{d+1\})\right) \qquad \forall T \subseteq \{1, 2, \ldots, d\}, \tag{5}$$

so the sign vector of Theorem 2 for the pivot $\{d+1\}$ determines the threshold function. Together with this reduction, the fact that $|\mathcal{T}| = 2^{\Theta(d^2)}$ implies an $\Omega\left(|M|^2\right)$ lower bound on the number of

comparison queries, even for zero-error randomized algorithms (Lemma 11): counting the cells of hyperplane arrangements gives the upper bound [54, 154], lower bounds of this order date back to 1965 [125, 157], and the asymptotics of this count have been determined [161, 99, 156].

We reduce the task of Theorem 2 to the problem studied in discrete geometry as *point-location in a hyperplane arrangement*, whose goal is to compute the labelling function $\ell_H : \mathbb{R}^m \ni x \mapsto (\operatorname{sign}\langle x, h\rangle)_{h\in H} \in \{-1,0,1\}^H$ for $m \in \{1,2,\ldots\}$ and a finite set $H \subset \mathbb{R}^m$; equivalently, one is asked to identify which polyhedral cell of the hyperplane arrangement induced by H contains the input $x \in \mathbb{R}^m$. Kane et al. [103] characterize the information cost of computing ℓ_H using a combinatorial structure of H, namely its *inference dimension* [101], a notion from active learning (Appendix C.4).

Lemma 4 ([103, Corollary 1.13])**.** *For any $m \in \{2,3,\ldots\}$, $b \in \{1,2,\ldots\}$, and any finite set $H \subset \{h \in \mathbb{Z}^m \mid \|h\|_\infty \le b\}$, there is a deterministic decision tree of depth $O(m \log(mb) \log(|H|+1))$ that computes ℓ_H and each of whose inner nodes asks either of the following:*

- $\operatorname{sign}(\langle x, h\rangle)$ *for the input $x \in \mathbb{R}^m$ and some $h \in H$.*
- $\operatorname{sign}(\langle x, h_1 - h_2\rangle)$ *for the input $x \in \mathbb{R}^m$ and some $h_1, h_2 \in H$.*

Applying Lemma 4 for $H \coloneqq \{\mathbf{1}_{M|S} - \mathbf{1}_{M|T} \mid T \subseteq M\}$ yields a decision tree that can be emulated by subset sum comparisons, hence Theorem 2. Furthermore, a similar argument turns Lemma 4 into another low-query algorithm that compares every subset with its complement. In Section 3, we leverage this subroutine for the special case with $|N| = 2$ agents, as in Theorem 8.

Theorem 3. *Given a finite set M with $|M| \ge 2$ and a queryable additive function $v : 2^M \to \mathbb{R}$, one can compute $(\operatorname{sign}(v(T) - v(M \setminus T)))_{T\subseteq M} \in \{-1,0,1\}^{2^M}$ using $O(|M|^2 \log|M|)$ comparison queries.*

Remark 4. The NP-hard problem PARTITION [104] can thus be solved with almost quadratically many comparison queries, as Kane et al. [103, Theorem 1.3] show for SUBSET-SUM, with comparisons of subset sums and a near-optimal number of queries. The bound of Theorem 3 is optimal up to a logarithmic factor, by a reduction as in Remark 3: for $(w,\theta) \in \mathbb{R}^{d+1}$, the additive function v on $M \coloneqq \{1,2,\ldots,d+1\}$ with $v(\{j\}) = -w_j/2$ for $j \le d$ and $v(\{d+1\}) = \frac{1}{2}\sum_{j=1}^d w_j - \theta$ satisfies $v(M\setminus T) - v(T) = \langle w, \mathbf{1}_{\{1,2,\ldots,d\}|T}\rangle - \theta$ for every $T \subseteq \{1,2,\ldots,d\}$. So the sign vector of Theorem 3 determines the threshold function, and computing it requires $\Omega(|M|^2)$ comparison queries.

Remark 5. Lemma 4 does not sort 2^M in the same way: for the vectors $\mathbf{1}_{M|S} - \mathbf{1}_{M|T}$ with disjoint $S, T \subseteq M$, its tree has depth $O(m^2 \log m)$, but its difference queries may mix coefficients ±1 and ±2, which are not comparison queries. Comparisons need not even determine such a query: all bundle values differ at $v = (3,1,5)$, yet $2v(\{a\}) = v(\{b,c\})$, so they do not determine whether $\{a\}$ is proportional for three agents, i.e., whether $v(\{a\}) \ge v(\{a,b,c\})/3$.

One pivot also settles other comparisons: for disjoint $A, B \subseteq M$ and every S with $A \subseteq S \subseteq M\setminus B$, additivity gives $v(A) - v(B) = v(S) - v((S\setminus A)\cup B)$. As $v(X) - v(Y) = v(X\setminus Y) - v(Y\setminus X)$, $O(2^w w \log|M|)$ greedily chosen pivots meeting every subcube $\{S \mid A \subseteq S \subseteq M\setminus B\}$ with $|A\cup B| \le w$ settle all comparisons of sets differing in at most w items (Appendix A.4), as Theorem 9 uses.

Theorem 4. *Let M be a finite set with $|M| \ge 2$, let $w \in \{1,2,\ldots\}$, and let*

$$\mathcal{B}_w \coloneqq \{(X,Y) \in 2^M \times 2^M \mid |X \triangle Y| \le w\}. \tag{6}$$

Given a queryable additive function $v : 2^M \to \mathbb{R}$, one can compute $(\operatorname{sign}(v(X) - v(Y)))_{(X,Y)\in\mathcal{B}_w} \in \{-1,0,1\}^{\mathcal{B}_w}$ using $O(w^2 2^w |M| \log^3|M|)$ comparison queries.

The exponential dependence on w is necessary: for $2 \le w \le |M|$ and an additive function that vanishes outside w items, this sign vector orders all bundles of these items, to which every comparison reduces, so it needs $f(w) \ge F_{w+1}$ comparisons even with zero error (Corollary 2).

2.5 Optimization and Selection

Cohen-Addad et al. [51, Theorem 1.1] find a minimizer of v over a family $\mathcal{F} \subseteq 2^M$ with a randomized algorithm that uses $O\left(|M| \log^2 |M| \log |\mathcal{F}|\right)$ comparison queries in expectation, and they note that the techniques of Kane et al. [103] can derandomize their query strategy for general linear optimization. Theorem 5 improves on this by a logarithmic factor, deterministically. Its proof replaces random samples by universal ones, as Kane et al. [103, Lemma 5.2] do for point location, and sorts each sample with the bound of Fredman [71].

Theorem 5. *Given a finite set M with $|M| \ge 2$, a nonempty family of subsets $\mathcal{F} \subseteq 2^M$, and a queryable additive function $v : 2^M \to \mathbb{R}$, one can find an element in $\arg\min_{S \in \mathcal{F}} v(S)$ using $O\left(|M| \log |M| \log\left(|\mathcal{F}| + 1\right)\right)$ comparison queries, each between two members of $\mathcal{F}$.*

Fixed pivots also support selection. Some thresholds, such as maximin shares, are bundle values that can be compared with any bundle once all bundles have been compared with it. A deterministic quickselect over bundles, whose pivots are medians of balanced samples, finds such a threshold.

Theorem 6. *Let M be a finite set with $|M| \ge 2$, and let τ map each additive function $v : 2^M \to \mathbb{R}$ to a value $\tau(v) \in \{v(T) \mid T \subseteq M\}$ such that, for each $S \subseteq M$, the sign of $\tau(v) - v(S)$ depends only on $\left(\operatorname{sign}\left(v(T) - v(S)\right)\right)_{T \subseteq M}$. Given a queryable additive function $v : 2^M \to \mathbb{R}$, one can find $S^* \subseteq M$ with $v\left(S^*\right) = \tau(v)$ and $\left(\operatorname{sign}\left(v(T) - \tau(v)\right)\right)_{T \subseteq M} \in \{-1, 0, 1\}^{2^M}$ using $O\left(|M|^3 \log |M|\right)$ comparison queries.*

When the thresholds are maximin shares with few parts, sieving the partitions as in Theorem 5 does better. For a finite set M, $d \in \{1, 2, \ldots\}$, and $v : 2^M \to \mathbb{R}$, we define $\mu_v(d) \coloneqq \max_{B \in \Pi_d(M)} \min_{j \in \{1,2,\ldots,d\}} v\left(B_j\right)$ (cf. Definition 9).

Theorem 7. *Given a finite set M with $|M| \ge 2$, an integer $d \ge 2$, and a queryable additive function $v : 2^M \to \mathbb{R}$, one can find $B \in \Pi_d(M)$ with $\min_j v\left(B_j\right) = \mu_v(d)$ and $\left(\operatorname{sign}\left(v(T) - \mu_v(d)\right)\right)_{T \subseteq M} \in \{-1, 0, 1\}^{2^M}$ using $O\left(|M|^2 \min\{d, |M|\} \log \min\{d, |M|\} \log |M|\right)$ comparison queries.*

2.6 Lower-Bound Principles

Our lower bounds rest on a few principles, which hold for every set of instances and also when *value queries*, each of which takes a pair $(i, S) \in N \times 2^M$ and returns $v_i(S) \in \mathbb{R}$, are allowed; we state and prove them in Appendix A.6. First, every lower bound against deterministic algorithms that are correct on a countable family of inputs, and every counting bound, holds for zero-error randomized algorithms (Lemma 11); in particular, the bound $E\left(\mathcal{P}\right) \le \mathrm{SR}_{\mathrm{rand}}(\mathcal{P})$ of Theorem 1 holds in this generality, and certificate bounds for $\mathcal{P}$ transfer to the search of every smaller predicate (Lemma 16). Second, with error probability $\varepsilon \in (0, 1/2)$, alternatives that differ from a baseline input in disjoint sets of queries still give a bound of about $(1 - 2\varepsilon)$ times their number (Lemma 12), whereas a large certificate alone does not; such alternatives arise from crossing the facets of a chamber (Lemma 13), from raising the values of disjoint bundles when valuations need not be additive, and, if only comparison queries are allowed, from flipping hidden signs of super-increasing magnitudes (Lemma 14). Third, if only comparison queries are allowed, verifying an allocation that is valid at several instances but invalid at each midpoint of two of them needs a logarithmic number

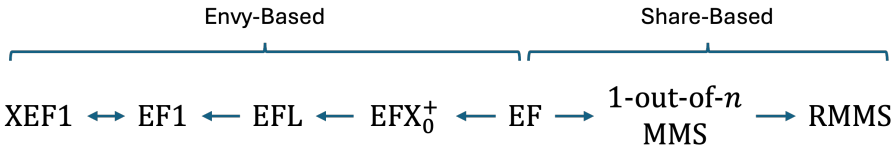


Figure 1: The hierarchy of fairness properties in monotone additive instances (Definitions 5 to 10).

of queries in their number (Lemma 15). For decision and verification, a majority vote lowers any error probability below $1/2$ to any positive constant at a constant-factor cost, so a lower bound for one error probability gives bounds of the same order for all others; for search this fails in general, and we state all our bounded-error lower bounds for each ε.

3 Query Complexity of Fairness and Efficiency Properties

Building on the tools developed in Section 2, we turn our attention back to the allocation of indivisible items and establish polynomial query-complexity guarantees for a variety of allocation properties. Among them, XEF1 + PO and IEF1 + PO allocations of any additive instance, and hence EF1 + PO allocations of goods and of chores, take polynomially many comparisons, although no polynomial-time algorithm is known (Theorem 29), and so do problems that are NP-hard or co-NP-hard with full information: finding an EF allocation for two agents or for identical valuations, finding an exact maximin share allocation, and verifying PO (Theorems 8, 21 and 24).

We supplement these guarantees by establishing polynomial lower bounds for fairness and efficiency properties in (monotone) additive instances, such as $\Omega(m)$ comparisons for PO on goods with two agents, against $O(\log m)$ for EF1 (Theorems 15 and 25), exponential lower bounds for EF with three or more agents (Theorem 14), and exponential lower bounds once we extend to the class of monotone *submodular* instances (Section 3.6). Sections 3.1 to 3.3 treat EF and its envy-based and share-based relaxations (Fig. 1 shows how they relate on monotone additive instances), Section 3.4 treats PO, Section 3.5 their combinations, and Section 3.6 non-additive valuations.

Throughout this section, $n \in \{2, 3, \ldots\}$ and $m \in \{2, 3, \ldots\}$ are arbitrary. For upper bounds, $\mathcal{C}$ denotes an arbitrary set of instances with at most n agents and at most m items unless otherwise specified; each lower bound specifies its set $\mathcal{C}$ of instances, and, by Definition 3, a lower bound on $\mathrm{Task}_\varepsilon(\mathcal{P})$ also bounds $\mathrm{Task}_{\mathrm{rand}}(\mathcal{P}) = \mathrm{Task}_0(\mathcal{P})$ and $\mathrm{Task}_{\mathrm{det}}(\mathcal{P})$. Many of our lower bounds hold even if value queries (Section 2.6) are allowed; the lower bounds for EF with two agents, with identical valuations, and with three or more agents (Theorems 12 to 14), however, hold only for comparison queries, which separates the two query models. As two value queries simulate a comparison, a lower bound of L value queries gives one of $L/2$ comparisons.

3.1 Envy-Freeness

We start by revisiting *envy-freeness (EF)*, one of the most classical and compelling fairness notions. For additive valuations, its comparison complexity depends sharply on the number of agents and on their valuations: verification needs $\Theta(n \min\{n, m\})$ comparisons, and search needs $\Theta(m^2)$ comparisons up to a logarithmic factor for two agents and polynomially many for identical valuations, but exponentially many in the worst case for three or more agents.

Definition 5 ([69, 152])**.** We define EF as the predicate on $\mathcal{C}$ that maps each instance $\mathcal{I} = (N, M, (v_i)_{i \in N}) \in \mathcal{C}$ to the set of allocations $A \in \mathcal{A}_0(\mathcal{I})$ such that $v_i(A_i) \geq v_i(A_j)$ for all $i, j \in N$.

3.1.1 Upper Bounds

Verifying EF needs no geometry: each agent compares her bundle with each other distinct bundle, and, on additive goods, an agent with an empty bundle needs only the comparison of M with $\emptyset$. Theorems 10 and 11 show that the following upper bounds are optimal up to constant factors.

Proposition 1. $\mathrm{VR}_{\mathrm{det}}(\mathrm{EF}) \leq n \min\{n-1, m\}$, *and* $\mathrm{VR}_{\mathrm{det}}(\mathrm{EF}) \leq s(s-1) + n - s$ *if* $\mathcal{C}$ *consists of monotone additive instances, where* $s \coloneqq \min\{n, m\}$.

Theorem 8 shows that polynomially many comparisons suffice for two agents and for identical valuations, despite the NP-hardness of the EF search problem by reduction from PARTITION [104]. For two agents, Theorem 3 lists the bundles that each agent values at least as much as their complements, and a similar test works for identical valuations with an even number of agents. For any number of agents with identical valuations, an EF allocation is a partition into bundles of equal value, so a maximin partition (Theorem 7) is EF whenever some allocation is, and selecting the threshold among small bundles (Theorem 6) is cheaper when there are many agents.

Theorem 8. $\mathrm{SR}_{\mathrm{det}}(\mathrm{EF}) = O\left(m^2 \log m\right)$ *if* $\mathcal{C}$ *consists of additive instances with two agents or of homogeneous additive instances with even numbers of agents. If* $\mathcal{C}$ *is the set of homogeneous additive instances with* n *agents and* m *items, then* $\mathrm{SR}_{\mathrm{det}}(\mathrm{EF}) \leq 2$ *if* $n > m$*; otherwise,*

$$\mathrm{SR}_{\mathrm{det}}(\mathrm{EF}) = O\left(\min\left\{nm^2 \log n \log m, \frac{m^3}{n} \log m \log(n+1)\right\}\right) \subseteq O\left(m^{2.5} \log^2 m\right). \tag{7}$$

For three or more heterogeneous agents, learning the order of all bundles for each agent (Corollary 2) and searching offline gives an exponential bound in terms of the largest number $f(m)$ of facets of a chamber, where $f(m) \leq B_m = \Theta\left(\vartheta^m m^{-\eta}\right)$ by Eq. (2) and Stirling's formula, with $\eta \coloneqq 1/\left(2 + 2\log_2\left((1-\lambda)/\lambda\right)\right) \approx 0.1807$.

Corollary 4. $\mathrm{SR}_{\mathrm{det}}(\mathrm{EF}) = O\left(nmf(m)\right) \subseteq O\left(nm^{1-\eta}\vartheta^m\right)$ *if* $\mathcal{C}$ *consists of additive instances, and* $\mathrm{SR}_{\mathrm{det}}(\mathrm{EF}) = O\left(n + \min\{n, m\} mf(m)\right)$ *if* $\mathcal{C}$ *consists of monotone additive instances.*

Near the diagonal, the exponent is the excess $m-n$ instead. If every agent values the whole set of items positively, every bundle of an EF allocation is nonempty, so any two bundles together contain at most $m-n+2$ items, and it suffices to learn the comparisons of such small bundles (Theorem 4); with identical positive values, the items of largest value even form bundles by themselves.

Theorem 9. *Let* $\mathcal{C}$ *consist of additive instances with* n *agents and* m *items in which* $v_i(M) > 0$ *for every agent* i*, and let* $r \coloneqq m-n$*. If* $m < n$*, then no instance in* $\mathcal{C}$ *has an EF allocation, and if* $m \geq n$*, then* $\mathrm{SR}_{\mathrm{det}}(\mathrm{EF}) = O\left(nm\left(r+2\right)^2 2^r \log^3 m\right)$*. If* $\mathcal{C}$ *consists of homogeneous additive instances whose item values are all nonnegative or all nonpositive and* $n \leq m < 2n$*, then* $\mathrm{SR}_{\mathrm{det}}(\mathrm{EF}) = O\left(m + (r+1)^2 \log(r+2)\right)$.

With Corollary 4, this gives $\min\{2^{m-n}, \vartheta^m\}\,\mathrm{poly}(n, m)$ on these instances, where the first term is smaller if $n > \lambda m \approx 0.227m$, and Theorem 14 shows that the exponential dependence on $m-n$ is necessary on monotone non-degenerate instances. For identical positive values, the last bound is optimal up to a logarithmic factor by Theorem 13.

3.1.2 Lower Bounds

We start by establishing quadratic lower bounds on query complexity of the EF allocation problems in additive instances, which match the verification upper bound in Proposition 1 that holds even for non-additive instances. Our construction is based on a linear-algebraic perturbation argument different from the one that Oh et al. [127] use to show an $\Omega(m)$ bound for deterministic algorithms and two agents, and our bounds hold for zero-error randomized algorithms, even with value queries; for bounded error, we flip hidden signs instead (Lemma 14).

Theorem 10. *For any $m \in \{2,3,\ldots\}$ and $n \in \{2,3,\ldots\}$, if $\mathcal{C}$ is the set of all additive instances with n agents and m items, then $\mathrm{SR}_{\mathrm{rand}}(\mathrm{EF}) = \Omega(n \min\{n,m\})$ and $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EF}) = \Omega(n \min\{n,m\})$. These bounds hold even if value queries are allowed. If only comparison queries are allowed, then $\mathrm{SR}_{\varepsilon}(\mathrm{EF})$ and $\mathrm{VR}_{\varepsilon}(\mathrm{EF})$ are at least $(1-2\varepsilon)\, n \min\{n-1,m\}$ for every $\varepsilon \in [0,1/2)$.*

A slight modification to the above linear-algebraic construction leads to the following lower bounds for monotone additive instances, which would become weaker than those of Theorem 10 only when $m = o(n)$, and the term n, which also bounds decision, comes from instances in which one or two agents value a single item.

Theorem 11. *For any $m \in \{2,3,\ldots\}$ and $n \in \{2,3,\ldots\}$, if $\mathcal{C}$ is the set of all monotone additive instances with n agents and m items, then $\mathrm{SR}_{\mathrm{rand}}(\mathrm{EF})$ and $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EF})$ are $\Omega\left(\min\left\{n^2, m^2\right\} + n\right)$, and $\mathrm{DC}_{\varepsilon}(\mathrm{EF})$ and $\mathrm{VR}_{\varepsilon}(\mathrm{EF})$ are at least $(1-2\varepsilon)(n-1)$ for every $\varepsilon \in [0,1/2)$. These bounds hold even if value queries are allowed. If only comparison queries are allowed, then $\mathrm{SR}_{\varepsilon}(\mathrm{EF})$ and $\mathrm{VR}_{\varepsilon}(\mathrm{EF})$ are at least $(1-2\varepsilon)(s(s-1)+n-s)$ for every $\varepsilon \in [0,1/2)$, where $s \coloneqq \min\{n,m\}$.*

With comparison queries, the verification bounds of Theorems 10 and 11 are exact: running the verifier of Proposition 1 with probability $1-2\varepsilon$, and otherwise answering uniformly at random without queries, shows that $\mathrm{VR}_{\varepsilon}(\mathrm{EF}) = (1-2\varepsilon)\, n \min\{n-1,m\}$ on additive instances and $\mathrm{VR}_{\varepsilon}(\mathrm{EF}) = (1-2\varepsilon)(s(s-1)+n-s)$ on monotone additive instances, where $s \coloneqq \min\{n,m\}$.

Comparison Queries Only. Without value queries, search is much harder. For two agents, Theorem 12 shows that a quadratic number of queries is needed, so Theorem 8 is optimal up to a logarithmic factor, whereas $2m$ value queries reveal both valuations. The proof reduces the knapsack problem to deciding EF and applies the lower bound of Grigoriev et al. [81] for randomized linear decision trees. For *promise search*, which may assume that an EF allocation exists, we write $\mathcal{C}^{\mathrm{EF}} \coloneqq \{\mathcal{I} \in \mathcal{C} \mid \mathrm{EF}(\mathcal{I}) \neq \emptyset\}$ and bound the search complexity with $\mathcal{C}^{\mathrm{EF}}$ in place of $\mathcal{C}$.

Theorem 12. *Let $m \geq 2$, and let $\mathcal{C}$ be the set of homogeneous additive instances with two agents and m items whose item values are all positive. Then $\mathrm{DC}_{\varepsilon}(\mathrm{EF}) = \Omega\left(m^2\right)$ and $\mathrm{SR}_{\varepsilon}(\mathrm{EF}) = \Omega\left(m^2\right)$ for every fixed $\varepsilon \in [0,1/2)$. These bounds also hold if $\mathcal{C}$ is instead the set of monotone non-degenerate additive instances with two agents and m items. If $m \geq 3$, the latter bound on $\mathrm{SR}_{\varepsilon}(\mathrm{EF})$ also holds with $\mathcal{C}^{\mathrm{EF}}$ in place of $\mathcal{C}$, for either set $\mathcal{C}$.*

For identical valuations, the items beyond $m-n+2$ can be forced into bundles of their own, so the quadratic term depends on $m-n$; by Theorems 8 and 9, the resulting bound is optimal up to a logarithmic factor when n is constant or $n \leq m < 2n$. With zero error, its leading constant is 1, by the strategy of Dobkin and Lipton [59] for the knapsack problem: the instances without an EF allocation fall into many regions, which a correct algorithm must tell apart, and the count of Zuev [162] and Kahn et al. [99] bounds their number.

Theorem 13. *Let $m \geq n$, and let $\mathcal{C}$ be the set of homogeneous additive instances with n agents and m items whose item values are all positive, or the set of those whose item values are all negative. Then, with $k \coloneqq m-n+2$, $\mathrm{DC}_{\mathrm{rand}}(\mathrm{EF})$ and $\mathrm{SR}_{\mathrm{rand}}(\mathrm{EF})$ are at least $\max\left\{n, k^2 - k\log_2 k - O(k)\right\}$, and $\mathrm{DC}_{\varepsilon}(\mathrm{EF})$ and $\mathrm{SR}_{\varepsilon}(\mathrm{EF})$ are $\Omega\left(k^2\right)$ for any fixed $\varepsilon \in [0, 1/2)$. If $m > n$, the latter bound on $\mathrm{SR}_{\varepsilon}(\mathrm{EF})$ also holds with $\mathcal{C}^{\mathrm{EF}}$ in place of $\mathcal{C}$.*

Remark 6. The term n of Theorem 13 does not extend to bounded error, which gives a natural separation from zero error. If $m = n$, an instance of Theorem 13 has an EF allocation if and only if all items have the same value. Every zero-error randomized algorithm that decides EF on these instances makes at least n comparison queries on an instance with an EF allocation, and n comparisons of single items around a cycle suffice. But for $\varepsilon \in (0, 1/2)$, repeating $\lceil \log_2(1/\varepsilon) \rceil$ times a comparison of $v(S)$ with $v(\pi(S))$ in both directions, for a uniformly random $S \subseteq M$ and a fixed cyclic permutation π of M, decides EF with one-sided error ε with $2\lceil \log_2(1/\varepsilon) \rceil$ comparisons, and returning any bijection between agents and items when it accepts solves search with the same error.

For three or more agents, search becomes exponentially hard, even on monotone non-degenerate instances. Let $F_1 \coloneqq F_2 \coloneqq 1$ and $F_{k+2} \coloneqq F_{k+1} + F_k$, so that $F_{k+2} \geq \varphi^k$. Two agents value a core of $m-n+1$ items by a point of a chamber of the ternary arrangement with F_{m-n+2} facets [47], up to tiny perturbations in opposite directions, and each other agent is satisfied only by a private item. Then an EF allocation exists exactly when the core valuation lies on a hyperplane whose normal involves a distinguished core item, as on each facet of this chamber, so every emptiness certificate at a point of the chamber must rule out each facet separately (Lemma 13).

Theorem 14. *Let $n \geq 3$, and let $\mathcal{C}$ be the set of monotone non-degenerate additive instances with n agents and $m \geq n+3$ items and $F \coloneqq F_{m-n+2}$, or the set of monotone additive instances with n agents and $m \geq 5$ items and $F \coloneqq F_m$. Then $E_{\emptyset}(\mathrm{EF}) \geq F$, and $\mathrm{DC}_{\varepsilon}(\mathrm{EF}) \geq (1-2\varepsilon)F$ and $\mathrm{SR}_{\varepsilon}(\mathrm{EF}) \geq (1-\varepsilon)F - \varepsilon$ for every $\varepsilon \in [0, 1/2)$. With $\mathcal{C}^{\mathrm{EF}}$ in place of $\mathcal{C}$, $\mathrm{SR}_{\mathrm{det}}(\mathrm{EF}) \geq F - 1$, and every randomized algorithm that returns an EF allocation with probability at least $p \in (0, 1]$ on every instance of $\mathcal{C}^{\mathrm{EF}}$ makes at least $\frac{(F+2)(pF-1)}{2F}$ queries in expectation on some instance of $\mathcal{C}^{\mathrm{EF}}$.*

So EF with three or more agents is exponentially harder than several closely related tasks, each of which polynomially many queries solve: verifying an EF allocation (Proposition 1), or certifying it, with the $n(n-1)$ comparisons of $v_i(A_i)$ with $v_i(A_j)$; finding an EF1 allocation of goods (Theorem 15), a PO allocation (Proposition 4), or even one that is XEF1 or IEF1 and PO (Theorem 29); finding an EF allocation for two agents, or for identical valuations (Theorem 8); and finding it with value queries, nm of which reveal the whole instance. By Lemma 16, the certificates bound promise search from below, but the last bound of Theorem 14 shows that they do not bound it from above: $\mathcal{C}^{\mathrm{EF}}$ is not a product of convex domains, so Theorem 1 does not apply to it.

Remark 7. The proof of Theorem 14 works for every chamber of the ternary arrangement on $m-n+1$ items, and one with $f(m-n+1)$ facets gives $\mathrm{DC}_{\mathrm{rand}}(\mathrm{EF}) \geq f(m-n+1)/2$ on monotone non-degenerate instances with $m \geq n+3$. So, for every fixed $n \geq 3$, $f(m-n+1)/2 \leq \mathrm{DC}_{\mathrm{rand}}(\mathrm{EF}) \leq \mathrm{SR}_{\mathrm{det}}(\mathrm{EF}) = O(mf(m))$ by Corollary 4. As EF allocations have certificates of size $n(n-1)$, Theorem 1 also relates these complexities to the emptiness certificates: for fixed n, $E_{\emptyset}(\mathrm{EF}) \leq \mathrm{DC}_{\mathrm{rand}}(\mathrm{EF}) \leq \mathrm{SR}_{\mathrm{det}}(\mathrm{EF}) = O(mE_{\emptyset}(\mathrm{EF}))$.

3.2 Envy-Based Relaxations

However, EF is often infeasible due to indivisibility of items. A large body of work has thus studied its approximations including *envy-freeness up to one item (EF1)*, which we define below along with its slightly stronger variant for monotone instances known as *envy-freeness up to one less-preferred*

good (EFL). An EF1 allocation exists in many relevant classes of instances [30, 28], including all monotone and antitone instances, and an EFL allocation exists in all monotone instances [65, 6]; both can be computed by low-query algorithms (Theorem 15).

Definition 6 ([110, 40, 18, 25])**.** We define EF1 as the predicate on $\mathcal{C}$ that maps each instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$ to the set of allocations $A \in \mathcal{A}_0(\mathcal{I})$ such that, for all $i, j \in N$, some $R \in \binom{M}{\leq 1}$ satisfies $v_i(A_i \setminus R) \geq v_i(A_j \setminus R)$. We define EFL as the predicate on $\mathcal{C}$ that maps each instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$ to the set of allocations $A \in \mathcal{A}_0(\mathcal{I})$ such that, for all $i \in N$ and all $j \in N$ with $|A_j| > 1$, some $g \in A_j$ satisfies $v_i(A_i) \geq \max\{v_i(\{g\}), v_i(A_j \setminus \{g\})\}$.

Remark 8. This form of EFL is that of Feige [65]; Barman et al. [25] also exempt the pairs for which A_j contains at most one item that agent i values positively, which makes a difference only if some items have value 0 to agent i.

3.2.1 Upper Bounds

For two agents with monotone valuations, Oh et al. [127] find an EF1 allocation with $O(\log m)$ queries by cut and choose, locating the cut by binary search, and Bu et al. [39, Lemma 2] do so with comparison queries for additive valuations of goods; $O(\log m)$ queries also suffice for three agents with monotone additive valuations [127, Theorem 4.5], [39, Theorem 6], and deterministic algorithms need $\Omega(\log m)$ comparisons even for a constant number of agents with identical valuations [39, Theorem 1]. Theorem 15 extends the two-agent bound to arbitrary valuations, for which the answers along the line need not be monotone, by bisecting between two cuts with different answers. For any number of agents, its bounds come from implementing envy-cycle elimination [110] with comparisons: the algorithm of Bhaskar et al. [30] for EF1 with doubly monotone or additive valuations, and that of Feige [65] for EFL with monotone valuations.

Adapting the notion of Bhaskar et al. [30] to sets of instances, we call a set $\mathcal{C}$ of instances *doubly monotone* if, for all sets N of agents and M of items, there are sets $G_i \subseteq M$ for $i \in N$ such that every instance $\left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$ satisfies, for each $i \in N$,

$$v_i(S \cup \{g\}) \geq v_i(S) \qquad \forall g \in G_i, \forall S \subseteq M \setminus \{g\}; \tag{8}$$

$$v_i(S \cup \{g\}) \leq v_i(S) \qquad \forall g \in M \setminus G_i, \forall S \subseteq M \setminus \{g\}; \tag{9}$$

that is, each agent i regards the items of G_i as goods and the other items as chores, in the same way in every instance of $\mathcal{C}$ with these agents and items. The set of all monotone instances and that of all antitone instances are doubly monotone, with $G_i = M$ and $G_i = \emptyset$, respectively.

Theorem 15. *If $\mathcal{C}$ is doubly monotone or consists of additive instances, then* $\mathrm{SR}_{\mathrm{det}}(\mathrm{EF1}) = O(nm \log \min\{n, m\})$. *Also,* $\mathrm{SR}_{\mathrm{det}}(\mathrm{EFL}) = O(nm \log \min\{n, m\})$ *if $\mathcal{C}$ consists of monotone instances. Moreover, if $\mathcal{C}$ consists of instances with two agents, then* $\mathrm{SR}_{\mathrm{det}}(\mathrm{EF1}) \leq \lceil \log_2(m-1) \rceil + 3$.

Envy-freeness up to the least-valued item (EFX) is another fairness notion celebrated as one of the most prominent relaxations of EF [41, 133], which we define in Definition 7 along with its variants allowing unallocated items and those concerning zero-valued items.

Definition 7 ([42, 18])**.** Let $c \in \{0, 1, \ldots\}$ be arbitrary. We define EFX_c as the predicate on $\mathcal{C}$ that maps each instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$ to the set of allocations $A \in \mathcal{A}_c(\mathcal{I})$ such that

$$v_i(A_i) \geq v_i(A_j \setminus \{g\}) \qquad \forall g \in M^+_{i,j}, \forall i, j \in N, \tag{10}$$

$$v_i(A_i \setminus \{g\}) \geq v_i(A_j) \qquad \forall g \in M^-_i, \forall i, j \in N, \tag{11}$$

where $M_{i,j}^{+} \coloneqq \{g \in A_j \mid v_i(A_j) > v_i(A_j \setminus \{g\})\}$ and $M_i^{-} \coloneqq \{g \in A_i \mid v_i(A_i) < v_i(A_i \setminus \{g\})\}$. The predicate EFX_c^+ on $\mathcal{C}$ is defined similarly except that equality is allowed in the definition of $M_{i,j}^{+}$.[2]

Proposition 2. *$\mathrm{VR}_{\mathrm{det}}(\mathcal{P}) = O(nm)$ for each $\mathcal{P} \in \{\mathrm{EF1}, \mathrm{EFL}\} \cup \{\mathrm{EFX}_c, \mathrm{EFX}_c^+ \mid c \in \{0, 1, \ldots\}\}$, and $\mathrm{VR}_{\mathrm{det}}(\mathrm{EFX}_0^+) = O(m \min\{n, m\} + n)$ if $\mathcal{C}$ consists of monotone additive instances. If $\mathcal{C}$ consists of antitone additive instances, then $\mathrm{VR}_{\mathrm{det}}(\mathrm{EFX}_0) \leq 2m + n^2$, and $\mathrm{VR}_{\mathrm{det}}(\mathrm{EFX}_0) \leq 2m + (n-2)\min\{n, (m-n)_+\}$ if they all have n agents and m items, where $(x)_+ \coloneqq \max\{x, 0\}$.*

Combining Proposition 2 with the existence results for monotone additive instances [44, 29] gives low search complexity of EFX by Corollary 1. In the case of $|N| = 2$ agents, we combine the cut-and-choose method with Theorem 32 to obtain a better query complexity that almost matches the $\Omega(|M|)$ lower bound from the value-query model [127].

Theorem 16. *If $\mathcal{C}$ consists of additive instances with two agents, then $\mathrm{SR}_{\mathrm{det}}(\mathrm{EFX}_0^+) = O(m \log m)$. If $\mathcal{C}$ consists of monotone additive instances with n agents, then $\mathrm{SR}_{\mathrm{det}}(\mathrm{EFX}_{n-3}^+) = O\left(m^3 / \log m\right)$ if $n \in \{3, 4\}$, and $\mathrm{SR}_{\mathrm{det}}(\mathrm{EFX}_{n-2}^+) = O\left(n^2 m^3 / \log(nm)\right)$.*

Recent work by Barman and Verma [24] introduces new envy-based notions, namely *extrospectively (resp. introspectively) envy-freeness up to one item (XEF1 (resp. IEF1))*, that coincide with EF1 in monotone (resp. antitone) instances under Definitions 6 and 8 and are compatible with Pareto optimality (PO), importantly for our simultaneous guarantee in Theorem 29.

Definition 8 ([24])**.** We define XEF1 as the predicate on $\mathcal{C}$ that maps each $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$ to the set of allocations $A \in \mathcal{A}_0(\mathcal{I})$ such that, for all $i, j \in N$, some $R \in \binom{M}{\leq 1}$ satisfies $v_i(A_i) \geq v_i(A_j \triangle R)$. Similarly, IEF1 is the predicate on $\mathcal{C}$ that maps each $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$ to the set of allocations $A \in \mathcal{A}_0(\mathcal{I})$ such that, for all $i, j \in N$, some $R \in \binom{M}{\leq 1}$ satisfies $v_i(A_i \triangle R) \geq v_i(A_j)$.

For IEF1, each agent finds a bundle that she values most among her own with at most one item toggled and compares it with each other distinct bundle, for any valuations. For XEF1, the item is toggled in the other bundle: on additive instances, removing one of its most valuable items or adding a least valuable outside item is best, and on monotone instances XEF1 coincides with EF1, whereas in general each toggle may need its own comparison.

Proposition 3. *$\mathrm{VR}_{\mathrm{det}}(\mathrm{IEF1}) = O(nm)$. Moreover, $\mathrm{VR}_{\mathrm{det}}(\mathrm{XEF1})$ is $O(nm)$ if $\mathcal{C}$ consists of additive instances or of monotone instances, and $O(nm \min\{n, m\})$ in general.*

3.2.2 Lower Bounds

We start with EF1; on goods, the bounds also hold for XEF1, which coincides with EF1 there, and the quadratic one for EFL and IEF1. Similarly to Theorem 10, a linear-algebraic construction gives quadratic lower bounds for monotone additive instances, matching the upper bound by Proposition 2 and Theorem 15 (up to a logarithmic factor) when $m = (1 + \Theta(1))\, n$. Moreover, this result is stronger than the known $\Omega(n \log(m/n))$ bound for deterministic algorithms with value queries [127] as long as $m = 2^{o(n)}$.

For verification with comparison queries, a family of instances that are invalid at all midpoints (Lemma 15) and a signed instance give more, so that $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EF1}) = \Theta(nm)$ if $m \leq n$, and $\Theta(m)$ for a constant number of agents if negative item values are allowed.

[2] For each $c \in \{0, 1, \ldots\}$, $\mathrm{EFX}_c^+(\mathcal{I})$ is slightly stronger than $\mathrm{EFX}_c(\mathcal{I})$: envy must also vanish after the removal of any item of zero marginal value. In addition, $\mathrm{EFX}_0^+(\mathcal{I}) \subseteq \mathrm{EFL}(\mathcal{I}) \subseteq \mathrm{EF1}(\mathcal{I})$ for any monotone instance $\mathcal{I}$.

Theorem 17. *Let $\mathcal{C}$ be the set of all monotone additive instances with n agents and m items. If $m \geq n+1$, then $\mathrm{SR}_{\mathrm{rand}}(\mathrm{EF1})$ and $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EF1})$ are $\Omega(n \min\{n, m-n\})$, even if value queries are allowed, and, if only comparison queries are allowed, $\mathrm{VR}_{\varepsilon}(\mathrm{EF1}) \geq (1-2\varepsilon)\, k\,(n-k)$ for every $\varepsilon \in [0, 1/2)$, where $k \coloneqq \min\{m-n, \lceil n/2 \rceil\}$. If only comparison queries are allowed, then $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EF1}) = \Omega(ns \log(1+m/s))$ with $s \coloneqq \min\{n, m\}$. The bounds above also hold for* XEF1*, and the bounds $\Omega(n \min\{n, m-n\})$ on $\mathrm{SR}_{\mathrm{rand}}$ and $\mathrm{VR}_{\mathrm{rand}}$ also hold for* EFL *and* IEF1*. If $\mathcal{C}$ is instead the set of all additive instances with n agents and m items, then $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EF1}) \geq m$, even if value queries are allowed.*

Again by a linear-algebraic construction, we derive lower bounds for the EFX allocation problems in monotone additive instances, which extend the $\Omega(m)$ bound known for the special case with $n = 2$ [127]; for verification, an instance with one bundle observed by many agents and the family of Theorem 17 show that Proposition 2 is optimal for all n and m, also for EFX_0^+; on chores, only the least costly among the costly own items and the best other bundle of each agent with two or more costly items matter, or, if some bundle is empty, whether an agent has two costly items, so verification takes $\Theta\left(m + n \min\left\{n, (m-n)_+\right\}\right)$ comparisons.

Theorem 18. *Let $\mathcal{C}$ be the set of all monotone additive instances with n agents and m items. If $m \geq n+1$, then $\mathrm{SR}_{\mathrm{rand}}(\mathrm{EFX}_0) = \Omega(n(m-n))$ and $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EFX}_0) \geq nm/4$, even if value queries are allowed, and, if only comparison queries are allowed, $\mathrm{VR}_{\varepsilon}(\mathrm{EFX}_0) \geq 2(1-2\varepsilon)\, k\,(n-k)$ for every $\varepsilon \in [0, 1/2)$, where $k \coloneqq \min\{m-n, \lceil n/2 \rceil\}$. If only comparison queries are allowed, then $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EFX}_0) \geq nm/4$ for all n and m, so that $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EFX}_0) = \Theta(nm)$. Moreover, $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EFX}_0^+) = \Theta(m \min\{n, m\} + n)$, even if value queries are allowed. If $\mathcal{C}$ is instead the set of all antitone additive instances with n agents and m items, then $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EFX}_0)$ is at least $m - n + r(n-1)$ with $r \coloneqq \min\{n, m-n\}$ if $m > n$, and at least $\lfloor m/2 \rfloor$ if $m \leq n$, even if value queries are allowed; so $\mathrm{VR}_{\mathrm{det}}(\mathrm{EFX}_0)$ and $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EFX}_0)$ are $\Theta\left(m + n \min\left\{n, (m-n)_+\right\}\right)$ there.*

Even when the monotone additive valuation functions are further assumed to be identical, we can obtain the $\Omega(m)$ lower bound on the search and verification complexity, which has been known only for the search problem in the special case with $n = 2$, for deterministic algorithms with value queries [127]; for verification, $m-1$ queries are needed and $2m+n-1$ comparisons suffice. Note that Plaut and Roughgarden [129] show an exponential lower bound for the EFX search problem in the class of homogeneous monotone submodular instances with two agents.

Theorem 19. *For any $n \in \{2, 3, \ldots\}$ and $m \in \{n+1, n+2, \ldots\}$, if $\mathcal{C}$ is the set of all homogeneous monotone additive instances with n agents and m items, then $\mathrm{SR}_{\mathrm{rand}}(\mathrm{EFX}_0) \geq R$, where $R \coloneqq n - 1 + \max\{qL, (n-1)(L-1)\}$ with $L \coloneqq \lfloor m/n \rfloor$ and $q \coloneqq m - nL$, so that $R = m-1$ if $m < 2n$ and $R \geq \lceil m/2 \rceil$, and $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EFX}_0) \geq m-1$. These bounds hold even if value queries are allowed, and the bound on search also holds for EFX_0^+, as $\emptyset \neq \mathrm{EFX}_0^+(\mathcal{I}) \subseteq \mathrm{EFX}_0(\mathcal{I})$ for every $\mathcal{I} \in \mathcal{C}$, by the proof of Theorem 32. Moreover, $\mathrm{VR}_{\mathrm{det}}(\mathrm{EFX}_0) \leq 2m+n-1$.*

3.3 Share-Based Fairness

As another relaxation of EF motivated by the classical cut-and-choose protocol, the *maximin share (MMS)* is one of the most celebrated share-based concepts in discrete fair division, which we define below in the generalized form of so-called ordinal approximation, with d parts instead of n.

Definition 9 ([40])**.** Let $d \in \{1, 2, \ldots\}$ be arbitrary. We define 1-out-of-d-MMS as the predicate on $\mathcal{C}$ that maps each instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$ to the set of complete allocations $A \in \mathcal{A}_0(\mathcal{I})$ such that $v_i(A_i) \geq \mu_{v_i}(d)$ for every $i \in N$, where the *maximin share* of a function $v : 2^M \to \mathbb{R}$ for d parts is, as for Theorem 7, $\mu_v(d) \coloneqq \max_{B \in \Pi_d(M)} \min_{j \in \{1,2,\ldots,d\}} v(B_j)$.

Algorithm 1 An algorithm for MMS verification.

Input: The sets of agents N and items M of an unknown additive instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$ in a set $\mathcal{C}$ of instances, an allocation $A \in \mathcal{A}(\mathcal{I})$, and $d \in \{1, 2, \ldots\}$.

Output: Either an assertion that $v_i(A_i) \geq \mu_{v_i}(d)$ for every $i \in N$ or $(i, A') \in N \times \Pi_d(M)$ such that $\min_{j=1}^{d} v_i(A'_j) > v_i(A_i)$.

$N' \leftarrow \{i^*\}$ for some $i^* \in \arg\min_{i \in N} v_1(A_i)$, found with comparisons of v_1, if $\mathcal{C}$ consists of homogeneous instances, and $N' \leftarrow N$ otherwise.
for all $i \in N'$ **do**
    $\mathcal{G}_i \leftarrow \{T \subseteq M \mid v_i(T) > v_i(A_i)\}$ using Theorem 2.
    for all $A' \in \Pi_d(M)$ **do**
        if $\left\{A'_j \mid j \in \{1, 2, \ldots, d\}\right\} \subseteq \mathcal{G}_i$ **then**
            return (i, A').
return an assertion that $v_i(A_i) \geq \mu_{v_i}(d)$ for every $i \in N$.

Building on Theorem 2, Algorithm 1 establishes Theorem 20, although it is not obvious from Definition 9 that polynomially many queries suffice: the bundles that agent i values above A_i, which Theorem 2 lists, determine offline whether some d-partition has all its parts among them, and, for identical valuations, one such list, for a bundle of least value, suffices.

Theorem 20. *Given an additive instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$, an allocation $A \in \mathcal{A}(\mathcal{I})$, and $d \in \{1, 2, \ldots\}$, one can verify that $v_i(A_i) \geq \mu_{v_i}(d)$ for every $i \in N$ or return $(i, A') \in N \times \Pi_d(M)$ such that $\min_{j=1}^{d} v_i(A'_j) > v_i(A_i)$ using $O\left(|N||M|^2 \log |M|\right)$ queries, and using $O\left(|M|^2 \log |M|\right)$ queries if $\mathcal{I}$ is homogeneous. Therefore, as completeness needs no queries, if $\mathcal{C}$ consists of additive instances, then $\mathrm{VR}_{\mathrm{det}}$(1-out-of-$d$-MMS) is $O\left(nm^2 \log m\right)$, which can be improved to $O\left(m^2 \log m\right)$ if they are homogeneous.*

For search, no existence result is needed. A maximin share $\mu_{v_i}(d)$ is a threshold of the kind that Theorem 6 selects, as $\mu_v(d) \geq v(S)$ if and only if some $B \in \Pi_d(M)$ has $v(B_j) \geq v(S)$ for every j, and similarly for strict inequalities. So Theorem 6, or Theorem 7 if d is small, compares every bundle with each agent's share, and an allocation that gives every agent a bundle worth at least her share, or the fact that none exists, is then found without further queries (Theorem 21). The MMS search problem is NP-hard by reduction from PARTITION [104]. For monotone instances with $|M|/2 < d < |M|$, some maximin partition makes a bundle of each item outside the $2(|M| - d)$ items of least value, so the share is the maximin share of these items for $|M| - d$ parts, capped at the least value of the other items (Lemma 22).

Theorem 21. *Let $d \in \{2, 3, \ldots\}$ be arbitrary. If $\mathcal{C}$ consists of additive instances, then*

$$\mathrm{SR}_{\mathrm{det}}(\text{1-out-of-}d\text{-MMS}) = O\left(n \min\left\{m^3 \log m, m^2 \min\{d, m\} \log \min\{d, m\} \log m\right\}\right). \tag{12}$$

If they are monotone and $m/2 < d \leq m$, then

$$\mathrm{SR}_{\mathrm{det}}(\text{1-out-of-}d\text{-MMS}) = O\left(n\left(m + (m-d)^3 \log(m - d + 2)\right)\right), \tag{13}$$

and $\mathrm{SR}_{\mathrm{det}}$(1-out-of-$d$-MMS) $= 0$ if they are monotone and $n(m - d + 1) \leq m$. If they are homogeneous, the factor n can be dropped from these bounds.

In particular, $\mathrm{SR}_{\mathrm{det}}(\text{1-out-of-2-MMS}) = O\left(m^2 \log m\right)$ for additive instances with two agents, and $\mathrm{SR}_{\mathrm{det}}(\text{1-out-of-}n\text{-MMS}) = O\left(\min\left\{m^3 \log m, m^2 \min\{n,m\} \log \min\{n,m\} \log m\right\}\right)$ for homogeneous additive instances. On monotone instances with $n(m-d+1) \leq m$, every complete allocation that gives every agent at least $|M|-d+1$ items qualifies, as every d-partition has a part inside any $|M|-d+1$ of these items.

Complete allocations in 1-out-of-d-MMS $(\mathcal{I})$ exist for every monotone additive instance $\mathcal{I}$ with $n \geq 2$ agents if $d = \min\{4\lceil n/3 \rceil, \lfloor 3n/2 \rfloor\}$ [93, 5], for some $d = n + O\left(\sqrt{n \log n}\right)$ [62], and, by a recent computer-checked proof, if $n = 4$ and $d = 5$ [140]; complete allocations in 1-out-of-$\left\lfloor \frac{9}{11} n \right\rfloor$-MMS $(\mathcal{I})$ exist for every antitone additive instance $\mathcal{I}$ with $n \geq 3$ agents [79]. Exact maximin share allocations exist even without additivity for every monotone or antitone instance with $n \geq 2$ agents and at most $n + c \log n$ items, for a constant $c > 0$, a threshold that is tight up to c for such instances [66].

Besides, Feige [65] shows that a relaxation of 1-out-of-$|N|$-MMS, namely the *residual maximin share (RMMS)* in Definition 10, can always be guaranteed for monotone valuations, together with EFL, and together with EFX if items may remain unallocated; this work has since been merged into Akrami et al. [6]. For monotone valuations, the RMMS is at least the *minimum EFX share (MXS)* of Caragiannis et al. [43]; for additive valuations, it is at least 2/3 of the MMS and, on non-degenerate instances, at least the strong EEFX share, the value from which on every bundle satisfies *epistemic EFX (EEFX)* [6, Observations 3.1 and 3.9 and Theorem 4.1]. So RMMS allocations satisfy these notions as well. Similarly to MMS, Theorem 2 allows us to establish polynomial query complexity of RMMS verification (Theorem 22), which we rely on to obtain simultaneous guarantees in Theorem 33, with EFL for complete allocations and with EFX for partial ones, as in the existence results of Feige [65] and Akrami et al. [6].

Definition 10 ([65, 6])**.** Let $c \in \{0, 1, \ldots\}$ be arbitrary. We define RMMS_c as the predicate on $\mathcal{C}$ that maps each instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$ to the set of allocations $A \in \mathcal{A}_c(\mathcal{I})$ such that $v_i(A_i) \geq \rho_{v_i}(|N|)$ for every $i \in N$, and let $\mathrm{RMMS} \coloneqq \mathrm{RMMS}_0$, where the *residual maximin share* $\rho_v(d) \in \{v(S) \mid S \subseteq M\}$ of a function $v : 2^M \to \mathbb{R}$ for $d \in \{1, 2, \ldots\}$ parts is defined as follows:[3]

$$\rho_v(d) \coloneqq \max\left\{ \tau \in \mathbb{R} \mid \begin{array}{c} \max\limits_{j \in \{1,2,\ldots,d-k\}} v(B_j) < \tau \;\Rightarrow \max\limits_{B' \in \Pi_k(M \setminus R)} \min\limits_{j \in \{1,2,\ldots,k\}} v\left(B'_j\right) \geq \tau \\ \forall B \in \Pi_{d-k}(R), \forall k \in \{1, 2, \ldots, d\}, \forall R \subseteq M \end{array} \right\}. \tag{14}$$

Theorem 22. *Given an additive instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$ and an allocation $A \in \mathcal{A}(\mathcal{I})$, one can verify that $v_i(A_i) \geq \rho_{v_i}(|N|)$ for every $i \in N$ or return $i \in N$ such that $\rho_{v_i}(|N|) > v_i(A_i)$ using $O\left(|N||M|^2 \log |M|\right)$ queries, and using $O\left(|M|^2 \log |M|\right)$ queries if $\mathcal{I}$ is homogeneous. In particular, if $\mathcal{C}$ consists of additive instances, $\mathrm{VR}_{\mathrm{det}}(\mathrm{RMMS}_c) = O\left(nm^2 \log m\right)$ for every $c \in \{0, 1, \ldots\}$, which can be improved to $O\left(m^2 \log m\right)$ if they are homogeneous.*

The residual share is a threshold of the same kind: by the footnote to Definition 10, $\rho_{v_i}(|N|) \geq v_i(S)$ if and only if, for every R, k, and B there, $v_i(B_j) \geq v_i(S)$ for some j or $\min_j v_i\left(B'_j\right) \geq v_i(S)$ for some $B' \in \Pi_k(M \setminus R)$, and similarly for strict inequalities. So Theorem 6 gives exact RMMS search as for MMS, without the existence results of Feige [65] and Akrami et al. [6] (Theorem 23); in fact, $\mathrm{RMMS}(\mathcal{I}) \neq \emptyset$ for every instance $\mathcal{I}$ with arbitrary valuations, by repeated envy-free matchings of one agent's residual partitions (Lemma 24), so the search never reports emptiness. On homogeneous instances, a maximin partition into $|N|$ parts is already an RMMS allocation, as $\rho \leq \mu$. For two agents with additive valuations, $\rho_{v_i}(2) = \mu_{v_i}(2)$ (Lemma 21), so Theorem 21

[3] We let $\max \emptyset = -\infty$ when $k = d$. For each $R \subseteq M$, $k \in \{1, 2, \ldots, d\}$, and $B \in \Pi_{d-k}(R)$, the condition in Eq. (14) holds if and only if $\tau \leq \max\left\{\max_{j \in \{1,2,\ldots,d-k\}} v(B_j), \max_{B' \in \Pi_k(M \setminus R)} \min_{j \in \{1,2,\ldots,k\}} v\left(B'_j\right)\right\}$, where the right-hand side belongs to $\{v(S) \mid S \subseteq M\}$ since $\Pi_k(M \setminus R) \neq \emptyset$. Hence the maximum in Eq. (14) is attained.

applies, and for monotone instances with $|M|/2 < |N| < |M|$, the $2(|M| - |N|)$ items of least value determine the residual share as they do the maximin share (Lemma 22).

Theorem 23. *If $\mathcal{C}$ consists of additive instances, then $\mathrm{SR}_{\det}(\mathrm{RMMS}) = O\left(nm^3 \log m\right)$, and $\mathrm{SR}_{\det}(\mathrm{RMMS}) = O\left(m^2 \log m\right)$ if, moreover, they have two agents. If $\mathcal{C}$ consists of homogeneous additive instances, then*

$$\mathrm{SR}_{\det}(\mathrm{RMMS}) = O\left(\min\left\{m^3 \log m, m^2 \min\{n, m\} \log \min\{n, m\} \log m\right\}\right). \tag{15}$$

If $\mathcal{C}$ consists of monotone additive instances with n agents and $m/2 < n < m$, then

$$\mathrm{SR}_{\det}(\mathrm{RMMS}) = O\left(n\left(m + (m-n)^3 \log(m-n+2)\right)\right), \tag{16}$$

where the factor n can be dropped if they are homogeneous, and $\mathrm{SR}_{\det}(\mathrm{RMMS}) = 0$ if they are monotone with $n \geq m$ agents.

Corollary 5 shows that, for a constant number of agents, the bounds of Theorem 21 for $d = n$, and those of Theorem 23 for two agents and for homogeneous instances, are optimal up to a logarithmic factor, even for randomized algorithms: on the homogeneous hard instances of Theorems 12 and 13, every RMMS allocation of an instance with an EF allocation is EF, so their lower bounds transfer with two more comparisons.

Corollary 5. *Let $m > n$ and $k \coloneqq m - n + 2$, let $\mathcal{C}$ be the set of homogeneous additive instances with n agents and m items whose item values are all positive, and let $\mathcal{P}$ be any predicate on $\mathcal{C}$ such that $\emptyset \neq \mathcal{P}(\mathcal{I}) \subseteq \mathrm{RMMS}_m(\mathcal{I})$ for every $\mathcal{I} \in \mathcal{C}$, such as 1-out-of-$n$-MMS, RMMS, EFL $\cap$ RMMS, $\mathrm{EFX}_m^+ \cap \mathrm{RMMS}_m$, $\mathrm{EFX}_0^+ \cap \mathrm{PO} \cap$ 1-out-of-n-MMS, and, if $n = 2$, $\mathrm{EFX}_0 \cap \mathrm{PO} \cap$ 1-out-of-2-MMS. Then $\mathrm{SR}_\varepsilon(\mathcal{P}) = \Omega\left(k^2\right)$ for every fixed $\varepsilon \in [0, 1/2)$.*

3.4 Efficiency

As an efficiency criterion, we adopt the well-studied economic notion of *Pareto optimality (PO)*. For additive instances, there always exists a PO allocation that can be computed by a low-query simple greedy algorithm (also in polynomial time). Finding one takes $\Theta(n + m \log n)$ comparisons on monotone additive instances and $\Theta(nm)$ on antitone instances and on all additive instances, also for randomized algorithms. Verifying that a given allocation is PO is harder: the problem is co-NP-hard, and, although polynomially many comparisons suffice, two agents already need $\Omega\left(m^2\right)$ of them, against m for finding a PO allocation.

Definition 11 ([128, 152])**.** For each instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$, an allocation $A' \in \mathcal{A}_0(\mathcal{I})$ is said to be a *Pareto improvement* on an allocation $A \in \mathcal{A}_0(\mathcal{I})$ if and only if $\emptyset \neq \{i \in N \mid v_i(A_i) < v_i(A'_i)\} \subseteq \{i \in N \mid v_i(A_i) \leq v_i(A'_i)\} = N$. We define PO as the predicate on $\mathcal{C}$ that maps each instance $\mathcal{I} \in \mathcal{C}$ to the set of allocations $A \in \mathcal{A}_0(\mathcal{I})$ on which there exists no Pareto improvement. Any allocation $A \in \mathrm{PO}(\mathcal{I})$ is called *Pareto optimal (PO)* in $\mathcal{I}$.

3.4.1 Upper Bounds

On monotone additive instances, comparing a bundle with the empty bundle tells whether it contains an item of positive value, so group testing finds the items that an agent values positively with few comparisons, and letting the agents take these items in turn gives a PO allocation (Algorithm 14). On homogeneous additive instances, no query is needed, as the bundle values of every complete allocation sum to the value of M, so that every complete allocation is PO.

Algorithm 2 An algorithm that verifies PO or returns a Pareto improvement.

Input: The sets of agents N and items M of an unknown additive instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$ and an allocation $A \in \mathcal{A}_0(\mathcal{I})$.

Output: Either an assertion that $A \in \mathrm{PO}(\mathcal{I})$ or a Pareto improvement on A.

1: **for all** $i \in N$ **do**
2: $\quad (\mathcal{L}_i, \mathcal{G}_i) \leftarrow (\{T \subseteq M \mid v_i(T) < v_i(A_i)\}, \{T \subseteq M \mid v_i(T) > v_i(A_i)\})$ using Theorem 2.
3: **for all** $A' \in \mathcal{A}_0(\mathcal{I})$ **do**
4: $\quad$ **if** $A'_i \notin \mathcal{L}_i$ for each $i \in N$ and $A'_{i'} \in \mathcal{G}_{i'}$ for some $i' \in N$ **then**
5: $\quad\quad$ **return** A' as a Pareto improvement on A.
6: **return** an assertion that $A \in \mathrm{PO}(\mathcal{I})$.

Proposition 4. *If $\mathcal{C}$ consists of additive instances, then $\mathrm{SR}_{\mathrm{det}}(\mathrm{PO}) = O(nm)$; if $\mathcal{C}$ consists of monotone additive instances, then $\mathrm{SR}_{\mathrm{det}}(\mathrm{PO}) = O(n + m \log n)$; and if $\mathcal{C}$ consists of homogeneous additive instances, then $\mathrm{SR}_{\mathrm{det}}(\mathrm{PO}) = 0$.*

In sharp contrast, verifying the Pareto optimality of a given allocation is a co-NP-hard problem even in the special case of two agents [33, 16]. Despite this computational hardness, we show that one can either verify that a given allocation is PO or find a Pareto improvement otherwise, only using a polynomial number of comparison queries. As demonstrated by Algorithm 2, this verification task boils down to the subset enumeration problem for which Theorem 2 ensures the polynomial query-complexity bound.

Theorem 24. *Given an additive instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$ and an allocation $A \in \mathcal{A}_0(\mathcal{I})$, one can verify that $A \in \mathrm{PO}(\mathcal{I})$ or return a Pareto improvement on A using $O\left(|N||M|^2 \log |M|\right)$ queries; in particular, $\mathrm{VR}_{\mathrm{det}}(\mathrm{PO}) = O\left(nm^2 \log m\right)$ if $\mathcal{C}$ consists of additive instances.*

3.4.2 Lower Bounds

Note that the PO search problem is total as any instance admits a solution (e.g., an allocation that maximizes the total value). On monotone additive instances, the n^m instances in which each item is valued at 1 by one agent and at 0 by the others have pairwise distinct unique PO allocations, which gives the counting bound of Theorem 25, and instances in which one agent values one item give its term n. So, for a constant number of agents, finding a PO allocation of goods needs $\Theta(m)$ comparisons by Proposition 4, more than finding an EF1 allocation, which needs only $O(\log m)$ comparisons for two agents (Theorem 15), but fewer than finding an EF allocation, which needs $\Omega\left(m^2\right)$ comparisons for two agents (Theorem 12).

Theorem 25. *For any $m \in \{2, 3, \ldots\}$ and $n \in \{2, 3, \ldots\}$, if $\mathcal{C}$ is the set of all monotone additive instances with n agents and m items, then $\mathrm{SR}_{\mathrm{rand}}(\mathrm{PO}) \geq \max\{m + n - 2, m \log_2 n\}$ and, for every $\varepsilon \in [0, 1/2)$, $\mathrm{SR}_{\varepsilon}(\mathrm{PO}) \geq \max\{(1-\varepsilon)n - 1, (1-\varepsilon)m \log_2 n - 1\}$. The bounds $m + n - 2$ and $(1-\varepsilon)n - 1$ hold even if value queries are allowed.*

With Proposition 4, PO search on monotone additive instances thus takes $\Theta(n + m \log n)$ comparisons, for deterministic and zero-error randomized algorithms and for every fixed $\varepsilon \in [0, 1/2)$. For two agents, $\mathrm{SR}_{\mathrm{det}}(\mathrm{PO}) = \mathrm{SR}_{\mathrm{rand}}(\mathrm{PO}) = m$ exactly: agent 1 compares each $\{g\}$ with $\emptyset$ and takes the items that she values positively, and agent 2 takes the rest, which gives each item to the first agent who values it positively, if any, as in the proof of Proposition 4. For verification, at the instance in which every agent values every item at 1, every allocation with $\min\{n, m\}$ nonempty

bundles is PO but needs m comparisons to each of their owners to be certified; for bounded error, we flip hidden signs instead (Lemma 14).

Theorem 26. *For any $m \in \{2,3,\ldots\}$ and $n \in \{2,3,\ldots\}$, if $\mathcal{C}$ is the set of all monotone additive instances with n agents and m items, then $\mathrm{VR}_{\mathrm{rand}}(\mathrm{PO}) \geq \max\{n-1, m\min\{n,m\}\}$ and $\mathrm{VR}_{\varepsilon}(\mathrm{PO}) \geq (1-2\varepsilon)(n-1)$ for every $\varepsilon \in [0,1/2)$, even if value queries are allowed. If only comparison queries are allowed, then $\mathrm{VR}_{\varepsilon}(\mathrm{PO}) \geq (1-2\varepsilon)\, m\min\{n,m\}/5$ for every $\varepsilon \in [0,1/2)$.*

These bounds are linear in m for a fixed number of agents, but, as for finding an EF allocation (Theorem 12), comparisons alone need quadratically many queries already for two agents, even without zero values or ties. If both agents value a core of $m-1$ items alike, and the remaining item at half the core up to tiny perturbations in opposite directions, then giving the core to agent 1 and the remaining item to agent 2 is PO exactly when the core does not split into two sets of equal value, so the arguments for Theorem 12 apply.

Theorem 27. *Let $\mathcal{C}$ be the set of monotone non-degenerate additive instances with n agents and m items, or the set of antitone non-degenerate additive instances with n agents and m items. If only comparison queries are allowed, then $\mathrm{VR}_{\varepsilon}(\mathrm{PO}) = \Omega\left(m^2\right)$ for every fixed $\varepsilon \in [0,1/2)$.*

With Theorems 26 and 28, this gives $\mathrm{VR}_{\varepsilon}(\mathrm{PO}) = \Omega\left(n+m^2\right)$ on monotone additive instances and $\Omega\left(nm+m^2\right)$ on antitone instances and on all additive instances, for every fixed $\varepsilon \in [0,1/2)$, so, with Theorem 24 and Remark 13, the comparison complexity of PO verification on each of these sets is determined up to a factor of $O(\min\{n,m\}\log m)$, which is logarithmic for every fixed n. On antitone instances and on all additive instances, Proposition 4 is optimal: at the instance in which every agent values every item at -1 (resp. 0), certifying any allocation needs m comparisons to each agent with an empty bundle (resp. to each agent), and, for bounded error, we set one value of super-increasing magnitude to 0 (Lemma 14).

Theorem 28. *For any $m \in \{2,3,\ldots\}$ and $n \in \{2,3,\ldots\}$, if $\mathcal{C}$ is the set of all antitone additive instances with n agents and m items, then $\mathrm{SR}_{\mathrm{rand}}(\mathrm{PO}) \geq (n-1)\, m$ and $\mathrm{VR}_{\mathrm{rand}}(\mathrm{PO}) \geq nm$, even if value queries are allowed, and, if only comparison queries are allowed, $\mathrm{SR}_{\varepsilon}(\mathrm{PO}) \geq ((1-\varepsilon)\, n-1)\, m$ and $\mathrm{VR}_{\varepsilon}(\mathrm{PO}) \geq (1-2\varepsilon)(n-1)\, m$ for every $\varepsilon \in [0,1/2)$. If $\mathcal{C}$ is instead the set of all additive instances with n agents and m items, then $\mathrm{SR}_{\mathrm{rand}}(\mathrm{PO}) \geq nm$ and $\mathrm{VR}_{\mathrm{rand}}(\mathrm{PO}) \geq nm$, even if value queries are allowed, and the above bounds on $\mathrm{SR}_{\varepsilon}(\mathrm{PO})$ and $\mathrm{VR}_{\varepsilon}(\mathrm{PO})$ hold as well.*

So PO search takes $\Theta(nm)$ comparisons on antitone instances and on all additive instances, for deterministic and zero-error randomized algorithms and for every fixed $\varepsilon \in [0,1/2)$, against $\Theta(n+m\log n)$ on monotone ones. On antitone instances, the zero-error bound is exact: the first $n-1$ agents in turn compare $\emptyset$ with $\{g\}$ for each remaining item g and take those that they value at 0, and agent n takes the rest, which gives each item to the first agent who values it at 0, if any, and so a PO allocation with at most $(n-1)\, m$ comparisons; hence $\mathrm{SR}_{\mathrm{det}}(\mathrm{PO}) = \mathrm{SR}_{\mathrm{rand}}(\mathrm{PO}) = (n-1)\, m$. The search lower bounds rely on zero values: if every agent values every item positively, or every item negatively, then the allocation that gives all items to one agent is PO, so no query is needed. Certificates (Definition 4) explain these bounds.

Remark 9. If $\mathcal{C}$ is the set of all monotone additive, all antitone additive, or all additive instances with n agents and m items, then $E(\mathrm{PO})$ is exactly $m+n-2$, $(n-1)\, m$, or nm, respectively. So the zero-error search bounds of Theorem 28 are the certificate sizes, whereas, on monotone additive instances, for each instance, some deterministic algorithm that is correct on all of $\mathcal{C}$ makes at most $m+n-2$ comparisons on it, so no argument at a single instance gives more than $m+n-2$, and the term $m\log_2 n$ of Theorem 25, which is larger if $3 \leq n \leq m$, needs counting. Rejections are cheap

to certify: if a complete allocation A is not PO in an instance, then some Pareto improvement on A changes the bundles of at most $\min\{n, m\}$ agents, and the comparisons of their bundles in the two allocations, one for each of these agents, show that it is a Pareto improvement on A in every instance that answers them alike.

3.5 Simultaneous Guarantees

Every additive instance admits XEF1 + PO and IEF1 + PO allocations [24], and Proposition 3 and Theorem 24 verify them with $O\left(nm^2 \log m\right)$ comparisons, so Corollary 1 gives polynomially many comparisons; learning first, for every agent, the comparisons of all bundles with $\emptyset$, and for IEF1 with a singleton of largest value, by Theorem 2 leaves only the at most $\min\{n, m\}$ agents with nonempty bundles to be certified, which makes the dependence on n linear (Theorem 29). As XEF1 and IEF1 coincide with EF1 on monotone and on antitone instances, respectively, this establishes the polynomial query complexity of computing an EF1 + PO allocation for any monotone or antitone additive instance.

Theorem 29. *Let $k \coloneqq \min\{n, m\}$. If $\mathcal{C}$ consists of additive instances, then $\mathrm{SR}_{\mathrm{det}}(\mathcal{P} \cap \mathrm{PO}) = O\left(nm^2 \log m + nkm^4 \log m / \log(km)\right) \subseteq O\left(nkm^4\right)$ for each $\mathcal{P} \in \{\mathrm{XEF1}, \mathrm{IEF1}\}$.*

On homogeneous instances, Theorem 32 does better, as $\mathrm{EFX}_0 \subseteq \mathrm{XEF1} \cap \mathrm{IEF1}$ on additive instances (Appendix B.5).

Remark 10. If $\mathcal{C}$ consists of antitone additive instances, each with at least as many agents as items, then $\mathrm{SR}_{\mathrm{det}}(\mathrm{EF1} \cap \mathrm{PO}) \leq 3nm$: after the comparisons of $\emptyset$ with every $\{g\}$ for every agent, each item that some agent values at 0 goes to such an agent, and the others, which every agent values negatively, go one each to distinct agents, by a bijection chosen lexicographically with ties kept (Appendix B.5).

The lower bounds for PO of Section 3.4 transfer to EF1 + PO and become stronger on goods: at the instance in which every agent values every item at 1, the bundle sizes of an EF1 allocation differ by at most one, so the owners of $\min\{n, m\}$ nonempty proper subsets of M each need m comparisons to certify PO (Lemma 25), and hidden flips (Lemma 14) at an instance with balanced blocks give a bounded-error form.

Theorem 30. *Let $r \coloneqq \min\{n, m\}$, and let $\mathcal{P}$ be $\mathrm{EF1} \cap \mathrm{PO}$, $\mathrm{XEF1} \cap \mathrm{PO}$, or $\mathrm{IEF1} \cap \mathrm{PO}$, or, if $n = 2$, $\mathrm{EFX}_0 \cap \mathrm{PO}$. If $\mathcal{C}$ is the set of all monotone additive instances with n agents and m items, then $\mathrm{SR}_{\mathrm{rand}}(\mathcal{P}) \geq \max\{n - 1, m \log_2 n, mr\}$ and, for every $\varepsilon \in [0, 1/2)$,*

$$\mathrm{SR}_{\varepsilon}(\mathcal{P}) \geq \max\left\{(1-\varepsilon)\, n - 1, (1-\varepsilon)\, m \log_2 n - 1, (1-2\varepsilon)\, mr/8\right\}, \tag{17}$$

where the last term also bounds $\mathrm{VR}_{\varepsilon}(\mathcal{P})$, and the bounds $n - 1$, mr, and $(1-\varepsilon)\, n - 1$ hold even if value queries are allowed. If $\mathcal{C}$ is instead the set of all antitone additive instances with n agents and m items, then $\mathrm{SR}_{\mathrm{rand}}(\mathcal{P}) \geq nm$, even if value queries are allowed, and, if only comparison queries are allowed, $\mathrm{SR}_{\varepsilon}(\mathcal{P}) \geq ((1-\varepsilon)\, n - 1)\, m$ for every $\varepsilon \in [0, 1/2)$.

Therefore, EF1 + PO search on monotone additive instances takes $\Omega\left(n + m \min\{n, m\}\right)$ comparisons for every fixed $\varepsilon \in [0, 1/2)$, against $\Theta\left(n + m \log n\right)$ for PO alone (Proposition 4 and Theorem 25), and $\Theta\left(nm\right)$ on antitone additive instances with $n \geq m$ (Remark 10); on all additive instances, the antitone bounds give $\Omega\left(nm\right)$ for XEF1 + PO and IEF1 + PO. On monotone instances, the proof also gives the terms $n - 1$ and $m \log_2 n$, with their bounded-error forms, for every predicate between $\mathrm{EF} \cap \mathrm{PO}$ and PO, and the last term of the display for every predicate between $\mathrm{EF} \cap \mathrm{PO}$ and $\mathrm{EF1} \cap \mathrm{PO}$, such as $\mathrm{EFL} \cap \mathrm{PO}$ and $\mathrm{EFX}_0 \cap \mathrm{PO}$ for any number of agents.

Remark 11. Let $r \coloneqq \min\{n, m\}$ and $\mathcal{P} \coloneqq \mathrm{EF1} \cap \mathrm{PO}$. The zero-error bounds $n-1$, mr, and nm of Theorem 30 are certificate sizes at the instances $\mathcal{J}_0$, $\mathcal{J}_1$, and $\mathcal{J}_{-1}$, in which every agent values each item at 0, 1, and -1, respectively: on all monotone additive instances with n agents and m items, every $A \in \mathcal{P}(\mathcal{J}_1)$ has least certificate size exactly mr, and the least size of a certificate of an allocation in $\mathcal{P}(\mathcal{J}_0)$ is $n-1$; on all antitone additive ones, every $A \in \mathcal{P}(\mathcal{J}_{-1})$ has least certificate size exactly nm. So, as for PO (Remark 9), no argument at these instances gives more.

For the stronger desiderata of EFX + PO, Lemma 5 extends the existence results of Plaut and Roughgarden [129] and Aziz [13] for goods to all additive instances with two agents. Maximizing the value of agent 1 subject to agent 2 being envy-free, and breaking ties in favor of agent 2, gives such an allocation, as Aziz [13] shows for goods, and it also gives both agents their maximin shares; Theorems 2, 3 and 5 compute it with almost quadratically many comparisons.

Lemma 5. *Let* $\mathcal{I} = (\{1,2\}, M, (v_1, v_2))$ *be any additive instance. Then* $\mathrm{EFX}_0(\mathcal{I}) \cap \mathrm{PO}(\mathcal{I}) \cap \text{1-out-of-2-MMS}(\mathcal{I}) \neq \emptyset$.

Theorem 31. *If* $\mathcal{C}$ *consists of additive instances with two agents, then* $\mathrm{SR}_{\mathrm{det}}(\mathrm{EFX}_0 \cap \mathrm{PO} \cap \text{1-out-of-2-MMS}) = O\left(m^2 \log m\right)$.

As 1-out-of-2-MMS = RMMS for two agents (Lemma 21), Corollary 5 shows that Theorem 31 is optimal up to a logarithmic factor; as $\mathrm{EFX}_0 \subseteq \mathrm{EF1} \cap \mathrm{XEF1} \cap \mathrm{IEF1}$ on additive instances, it also finds EF1 + PO, XEF1 + PO, and IEF1 + PO allocations for two agents with $O\left(m^2 \log m\right)$ comparisons, against $\Omega(m)$ by Theorem 30.

Lemma 5 no longer holds if EFX_0 is replaced with EFX_0^+, which rules out removal of zero-valued items: Plaut and Roughgarden [129, Theorem 5.1] give a monotone additive instance with two agents where no EFX_0^+ allocation is PO. Nor can PO be strengthened to *fractional Pareto optimality (fPO)*, which also rules out Pareto improvements by fractional allocations, since Garg et al. [77, Theorem 2] give an antitone additive instance with two agents that admits no allocation that is both EFX_0 and fPO. In fact, comparison queries cannot find EF1 + fPO allocations in general: if agents 1 and 2 value two goods at $(3,1)$ and $(2,1)$, respectively, or the other way around, every comparison is answered alike, but EF1 gives each agent one good, and fPO then gives the first good to agent 1 in the former case and to agent 2 in the latter. Nor does Lemma 5 extend to four agents: Mahabadi et al. [118, Theorem 1] give an instance with four agents, eight goods, and positive additive values in which no allocation is both EFX_0 and PO; for chores, He and Tao [85, Theorem 2] give one with n agents for every $n \geq 4$, with all values in $\{-r, -1\}$ for some $r > 1$.

In addition, we show that existing algorithms for computing an EFX + PO allocation in any homogeneous additive instance [8, 15] can be emulated only using almost linear comparison queries, and so can that of Aleksandrov and Walsh [8] for *absolutely identical* valuations, $|v_i(\{g\})| = |v_j(\{g\})|$ for all agents i and j and items g, with $O(nm + m \log m)$ comparisons, which is optimal if $n \geq \log m$ (Proposition 6). As moving items that every agent values at 0 changes no utility, and so keeps PO, moving them into a bundle of least value then also gives EFX_0^+. A partition that maximizes its sorted vector of bundle values lexicographically is moreover EFX_0 and gives every agent her maximin share, as is known for goods [129]; for mixed items with identical valuations, Chen and Liu [46] show that a leximin allocation is EFX, and Hosseini et al. [94] that EFX and MMS allocations exist. So exact maximin shares can be added at the cost of learning the maximin share and of a certificate search over the valuations consistent with it (Corollary 1).

Theorem 32. *If* $\mathcal{C}$ *consists of homogeneous additive instances, then* $\mathrm{SR}_{\mathrm{det}}(\mathcal{P} \cap \mathrm{PO}) = O(m \log m)$ *for each* $\mathcal{P} \in \left\{\mathrm{EFX}_0, \mathrm{EFX}_0^+\right\}$. *Moreover, with* $k \coloneqq \min\{n, m\}$,

$$\mathrm{SR}_{\mathrm{det}}(\mathrm{EFX}_0^+ \cap \mathrm{PO} \cap \text{1-out-of-}n\text{-MMS}) = O\left(m^3 / \log m + \min\left\{m^3 \log m, m^2 k \log k \log m\right\}\right). \quad (18)$$

As $\mu_v(n) \geq \rho_v(n)$ (Lemma 21), this gives complete $\mathrm{EFX}_0^+ + \mathrm{RMMS}$ allocations on homogeneous instances, against possibly partial ones in Theorem 33, and Corollary 5 gives the lower bound $\Omega\left((m-n)^2\right)$.

Lastly, combining the verification bounds by Proposition 2 and Theorem 22 with the existence results by Feige [65] and Akrami et al. [6], Corollary 1 gives a polynomial-query algorithm for computing an allocation that simultaneously satisfies envy-based and share-based fairness criteria as follows: a complete EFL allocation, or a possibly partial EFX_m^+ allocation, that gives every agent her residual maximin share.

Theorem 33. *If $\mathcal{C}$ consists of monotone additive instances, then, for each $\mathcal{P} \in \left\{\mathrm{EFL}, \mathrm{EFX}_m^+\right\}$, we have $\mathrm{SR}_{\det}(\mathcal{P} \cap \mathrm{RMMS}_m) = O\left(n^2 m^4 \log m / \log(nm)\right)$. If they have two agents, then even $\mathrm{SR}_{\det}(\mathrm{EFX}_0^+ \cap \text{1-out-of-2-MMS}) = O\left(m^2 \log m\right)$, where this predicate is contained in both $\mathrm{EFL} \cap \mathrm{RMMS}_m$ and $\mathrm{EFX}_m^+ \cap \mathrm{RMMS}_m$. If $\mathcal{C}$ consists of homogeneous monotone additive instances, then, with $k \coloneqq \min\{n, m\}$,*

$$\mathrm{SR}_{\det}(\mathcal{P} \cap \mathrm{RMMS}_m) = O\left(\min\left\{m^3 \log m, m^2 k \log k \log m\right\}\right) \tag{19}$$

for each such $\mathcal{P}$, and this bound also holds for $\mathrm{EFL} \cap$ 1-out-of-n-MMS.

The algorithms for two agents and for homogeneous instances even give every agent her maximin share, which is at least her residual maximin share (Lemma 21), and, with Corollary 5, they are optimal up to a logarithmic factor for a constant number of agents.

Ordinal approximations of MMS are compatible with envy-based guarantees as well, at least on *ordered* instances, in which all agents rank the items alike: Akrami and Reichert [4] show the existence of allocations that are 1-out-of-4 $\lceil n/3 \rceil$-MMS and EF1, and of allocations that are 1-out-of-$\lceil 3n/2 \rceil$-MMS and EFX.

3.6 Beyond Additive Valuations

This subsection shows that exponential lower bounds hold for monotone *submodular* instances.

Definition 12. An instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$ is said to be *submodular* if v_i is submodular, i.e., $v_i(X) + v_i(Y) \geq v_i(X \cup Y) + v_i(X \cap Y)$ for any $X, Y \subseteq M$, for each $i \in N$.

Deciding whether an EF allocation exists then requires exponentially many queries in $m - n$, even if value queries are allowed and the instance is homogeneous; if $m < n$, every complete allocation has an empty bundle, so, by monotonicity and submodularity, a homogeneous instance with valuation v has an EF allocation if and only if $v(M) = v(\emptyset)$, which one comparison decides. For $m \geq n$, $n-2$ items are each worth as much as M, and every bundle of exactly $\lfloor k/2 \rfloor$ of the other $k \coloneqq m - n + 2$ items is penalized by $1/2$, except, if k is even, those that contain a fixed item, so that no EF allocation exists; lifting the penalty of one such bundle creates one, and each query detects at most one such lifting (Appendix A.6). If $m > n$, a valuation that counts the items of a bundle among $n+1$ fixed ones, up to 2, admits no EF allocation either, and counting a pair of them as one item creates one, which gives a bound quadratic in n.

Theorem 34. *For any $n \in \{2, 3, \ldots\}$ and $m \in \{n, n+1, \ldots\}$, let $k \coloneqq m - n + 2$, and let $L \coloneqq \binom{k}{k/2}/2$ if k is even and $L \coloneqq \binom{k}{\lfloor k/2 \rfloor}$ if k is odd, so that $L \geq 2^{k-1}/(k+1)$. If $\mathcal{C}$ is the set of all homogeneous monotone submodular instances with n agents and m items, then $\mathrm{DC}_{\mathrm{rand}}(\mathcal{P}) \geq \max\{n, L\}$ and, for every $\varepsilon \in [0, 1/2)$, $\mathrm{DC}_{\varepsilon}(\mathcal{P}) \geq (1 - 2\varepsilon)\max\{n, L\}$ for every predicate $\mathcal{P}$ on $\mathcal{C}$ such that $(\mathrm{EF} \cap \mathrm{PO})(\mathcal{I}) \subseteq \mathcal{P}(\mathcal{I}) \subseteq \mathrm{EF}(\mathcal{I})$ for every $\mathcal{I} \in \mathcal{C}$, such as EF and $\mathrm{EF} \cap \mathrm{PO}$. If, moreover, $m > n$, then $\mathrm{DC}_{\mathrm{rand}}(\mathrm{EF}) \geq \binom{n+1}{2}$ and $\mathrm{DC}_{\varepsilon}(\mathrm{EF}) \geq (1 - 2\varepsilon)\binom{n+1}{2}$. All these bounds also hold for $\mathrm{SR}_{\mathrm{rand}}$ and $\mathrm{SR}_{\varepsilon}$, even if value queries are allowed.*

For $n = 2$, Theorem 34 recovers, up to a factor of 2 and with submodular hard instances, the bound $\binom{m}{m/2}$ of Oh et al. [127, Proposition 3.3] for deterministic algorithms with value queries on two agents with identical monotone valuations and an even number m of items.

Furthermore, finding or verifying a PO allocation requires exponentially many queries, even if value queries are allowed: if agent 1 values each bundle X at $\min\{|X|, \lfloor m/2 \rfloor\}$, minus 1/2 if $|X| = \lfloor m/2 \rfloor$ and $X \neq Y$ for a hidden bundle Y, agent 2 at $\min\{|X|, \lceil m/2 \rceil\}$, and the others at 0, then $(Y, M \setminus Y, \emptyset, \ldots, \emptyset)$ is the only PO allocation, and each query detects at most one hidden bundle. For verification, it suffices that all agents value bundles alike, with the penalty at the bundles of $\lceil m/2 \rceil$ items: the allocation that gives $\lceil m/2 \rceil + 1$ items to agent 1 and the others to agent 2 is PO if no bundle is hidden, but not if one is.

Theorem 35. *For any $m \in \{2, 3, \ldots\}$ and $n \in \{2, 3, \ldots\}$, let $\mathcal{C}$ be the set of all monotone submodular instances with n agents and m items, and let $K \coloneqq \binom{m}{\lfloor m/2 \rfloor}$. Then $\mathrm{SR}_{\mathrm{rand}}(\mathrm{PO}) \geq \max\{n-1, K-1\}$, $\mathrm{VR}_{\mathrm{rand}}(\mathrm{PO}) \geq \max\{n-1, K\}$, and, for every $\varepsilon \in [0, 1/2)$,*

$$\mathrm{SR}_{\varepsilon}(\mathrm{PO}) \geq \max\{(1-\varepsilon)\, n - 1, (1-\varepsilon)\, K - 1\}, \qquad \mathrm{VR}_{\varepsilon}(\mathrm{PO}) \geq (1-2\varepsilon) \max\{n-1, K\}. \tag{20}$$

The bounds $K - 1$ and $(1-\varepsilon)\, K - 1$ on search also hold with PO *replaced by any predicate $\mathcal{P}$ on $\mathcal{C}$ such that, for every $\mathcal{I} \in \mathcal{C}$, $\mathcal{P}(\mathcal{I}) \subseteq \mathrm{PO}(\mathcal{I})$ and $\mathcal{P}(\mathcal{I})$ contains every complete allocation that gives every agent her largest value, such as* EF $\cap$ PO *and* PO $\cap$ 1-out-of-n-MMS*, and the bounds K and $(1-2\varepsilon)\, K$ on verification also hold if $\mathcal{C}$ is the set of all homogeneous monotone submodular instances with n agents and m items. These bounds hold even if value queries are allowed.*

Remark 12. On additive instances, PO search and verification take $O(nm)$ and $O(nm^2 \log m)$ comparisons (Proposition 4 and Theorem 24), and no queries if the instance is homogeneous, as every complete allocation is then PO; so Theorem 35 separates monotone submodular valuations from additive ones for both tasks. Homogeneity does not help for monotone submodular valuations. PO verification needs K queries, with K as in Theorem 35, even on homogeneous instances with $m \leq n$, where giving each item to a different agent is PO. At the instances in the proof of Theorem 34 that have an EF allocation, the PO allocations, and also the 1-out-of-n-MMS allocations, which polynomially many comparisons find and verify on homogeneous additive instances (Theorems 20 and 21), are exactly those that give every agent the largest value, and each of them reveals which penalty was lifted. So, for $m \geq n$, if $\mathcal{C}$ is the set of all homogeneous monotone submodular instances with n agents and m items, L is as in Theorem 34, and $\mathcal{P} \in \{\mathrm{PO}, \text{1-out-of-}n\text{-MMS}\}$, then $\mathrm{SR}_{\mathrm{rand}}(\mathcal{P}) \geq L - 1$ and $\mathrm{VR}_{\mathrm{rand}}(\text{1-out-of-}n\text{-MMS}) \geq L - 1$, and $\mathrm{SR}_{\varepsilon}(\mathcal{P}) \geq (1-\varepsilon)\, L - 1$ and $\mathrm{VR}_{\varepsilon}(\text{1-out-of-}n\text{-MMS}) \geq (1-2\varepsilon)\,(L-1)$ for every $\varepsilon \in [0, 1/2)$, even if value queries are allowed.

These bounds are nearly tight for few agents, and for few more items than agents. For a monotone valuation v_i and a bundle R, the indicator functions of $v_i(X) \geq v_i(R)$ and of $v_i(X) > v_i(R)$ are monotone in X, and each of their values is a comparison, so Hansel's theorem [84], by which a monotone Boolean function on m variables can be learned with at most $2\binom{m}{\lfloor m/2 \rfloor}$ queries of its values [see 12], gives a verifier for PO, and a search algorithm that maximizes the values of the agents one after another. For EF, the values of the pairs that contain an item of largest value determine which bundles that contain such an item have that value, and only small bundles of the other items matter.

Proposition 5. *For any $m \in \{2, 3, \ldots\}$ and $n \in \{2, 3, \ldots\}$, let $K \coloneqq \binom{m}{\lfloor m/2 \rfloor}$. If $\mathcal{C}$ is the set of all monotone instances with n agents and m items, then $\mathrm{SR}_{\mathrm{det}}(\mathrm{PO}) \leq 3\,(n-1)\, K$ and $\mathrm{VR}_{\mathrm{det}}(\mathrm{PO}) \leq 4nK$; if $\mathcal{C}$ is the set of all monotone submodular ones, then $\mathrm{SR}_{\mathrm{det}}(\mathrm{PO}) \leq n + 3\min\{n, m\}\, K$ and $\mathrm{VR}_{\mathrm{det}}(\mathrm{PO}) \leq n + 4\min\{n, m\}\, K$. If $\mathcal{C}$ is the set of all homogeneous monotone submodular instances*

with n agents and m items with $m < 2n$, then $\mathrm{DC}_{\mathrm{det}}(\mathrm{EF})$ *and* $\mathrm{SR}_{\mathrm{det}}(\mathrm{EF})$ *are* $O\left(m^2 + \varphi^{2(m-n)}\right)$, *where* $\varphi \coloneqq \left(1+\sqrt{5}\right)/2 < 1.62$.

Therefore, for any fixed n and $\varepsilon \in [0, 1/2)$, PO search and verification on monotone submodular instances take $\Theta\left(\binom{m}{\lfloor m/2 \rfloor}\right)$ comparisons for deterministic, randomized, and bounded-error algorithms alike, by Theorem 35, and, for any fixed $s \geq 1$, EF decision and search on homogeneous monotone submodular instances with $m = n + s$ take $\Theta\left(m^2\right)$ comparisons, by Theorem 34.

4 Discussion

Total Runtime. PO verification, MMS search, and EF search for two agents or for identical valuations take polynomially many comparisons (Theorems 8, 21 and 24), although they are NP-hard or co-NP-hard [104, 33, 16]; as the values answer a comparison with $O(m)$ additions, this hardness transfers, so that, unless P = NP, no comparison-query algorithm solves them in polynomial time. Whether polynomially many queries and polynomial time can be achieved together is open, e.g., for EF1 + PO with two agents, where the values give an EF1 + PO allocation in polynomial time [18, Theorem 2], so that no hardness transfers. A cutting-plane method with a verifier as its separation oracle, as Cohen-Addad et al. [51] use for one linear objective, would need a polynomial-time verifier for a predicate within EF1 + PO that is nonempty on every instance; devising one is the crux, as verifying PO is co-NP-hard. When the number of agents is not fixed, no polynomial-time algorithm is known for EF1 + PO even with full information, on goods or on chores [26, 9, 119].

Inter-Query Runtime. Our algorithms choose each query by computations on the previous answers that may take exponential time: the majority vote of Theorem 1 keeps up to exponentially many consistent types, and the point location of Kane et al. [103] behind Theorem 2 checks which of the remaining bundles the answers determine, which Kane et al. [101] leave open to do in time polynomial in m even for one threshold function. Long delays between queries can disrupt the interaction with agents even when few queries suffice, which is why adaptive questionnaires choose the next comparison without noticeable delay [149, 83]. It would therefore be relevant to bound the *inter-query runtime* of algorithms, as Cohen-Addad et al. [51] and Fox and Woodruff [70] do for specific problems; with the computation after the last query included, this is impossible for the NP-hard tasks above unless P = NP, but it may be possible, e.g., for EF1 + PO with two agents.

Conditional Implications. By Corollary 1, existence on a domain of Theorem 1 turns every zero-error polynomial-query verifier into a polynomial-query search, as in Theorems 16, 29 and 33, however many constraints of whatever bit size define the domain, as they are never queried; for example:

Corollary 6. *Let $\mathcal{C}$ consist of the additive instances with agents N and items M such that $v_i \in D_i$ for every $i \in N$, where each $D_i \subseteq \mathbb{R}^M$ is as in Theorem 1, e.g., a nonempty convex set.*

(a) If $\mathrm{EF1}(\mathcal{I}) \cap \mathrm{PO}(\mathcal{I}) \neq \emptyset$ *for every* $\mathcal{I} \in \mathcal{C}$, *then* $\mathrm{SR}_{\mathrm{det}}(\mathrm{EF1} \cap \mathrm{PO}) = O\left(n^2 m^4 \log m / \log(nm)\right)$.

(b) If $\mathrm{EFX}_0(\mathcal{I}) \neq \emptyset$ *for every* $\mathcal{I} \in \mathcal{C}$, *then* $\mathrm{SR}_{\mathrm{det}}(\mathrm{EFX}_0) = O\left(n^2 m^3 / \log(nm)\right)$.

Indeed, the verifiers of Proposition 2 and Theorem 24 verify EF1 ∩ PO with $O\left(nm^2 \log m\right)$ comparisons and EFX_0 with $O(nm)$, so Corollary 1 gives both bounds. The bounds also hold if the profiles $(v_i)_{i \in N}$ range over any convex subset of $\left(\mathbb{R}^M\right)^N$, coupling the agents: Helly's theorem

in its affine hull, of dimension $d \leq nm$, refutes every unrealizable vector of answers with at most $\max\{2, d+1\}$ of them (cf. Lemma 1), replacing $m+1$ in the proof of Theorem 1.

The assumption of (a) holds for goods, chores, and two agents [42, 120], [18, Theorem 2], but is open for additive instances with three or more agents [24, Section 1.1], for which EF1 + fPO can fail [117, Theorem 4.3]. That of (b) holds for two agents (Lemma 5), for goods with three agents [44], and for goods with any number of agents and at most three distinct valuations [96], is open for goods with four or more agents in general and for chores with three agents, and fails for chores with every $n \geq 4$, even if all values are negative [85, Theorem 1]; earlier counterexamples for chores needed non-additive costs, such as superadditive ones [49, Theorem 1]. If it is known which agents share a valuation, the profiles of goods with at most three distinct valuations shared in this way form a convex set, so the bound of (b) applies to them. Beyond additive valuations, these existence results fail already for submodular valuations: an instance with two agents and eight goods has no allocation that is both EF1 and PO [117, Theorem 1.1], and instances with $n \geq 3$ agents and $m \geq n+5$ goods have no EFX allocation [7, Theorem 2], [115, Theorem 1.1]. Where existence holds beyond additivity, the proof of Theorem 1 still applies, as it needs only finitely many types: for all (monotone) set functions, three answers refute every unrealizable vector of answers, but the $2^{\Theta(nm2^m)}$ types make the bound no better than sorting all bundles; additivity matters through the type count.

Open Questions:

EF1 + PO. On goods with $2 \leq n \leq m$, is $E(\mathrm{EF1} \cap \mathrm{PO}) = O(nm)$? By Theorem 1, this would give $\mathrm{SR}_{\mathrm{det}}(\mathrm{EF1} \cap \mathrm{PO}) = O\left(n^2 m^3 / \log(nm)\right)$, against $O\left(n^2 m^4\right)$ (Theorem 29) and $\Omega(nm)$ (Theorem 30); certificates of size $O(nm)$ on all additive instances with $n \leq m$ would do the same for XEF1 + PO and IEF1 + PO. Conversely, as $E(\mathrm{EF1} \cap \mathrm{PO}) = O\left(nm^2 \log m\right)$ (Proposition 2 and Theorem 24), no lower-bound argument at a single instance gives more, nor does counting instances with disjoint sets of valid outputs give more than $m \log_2 n$, as there are n^m complete allocations. With two agents on goods, finding an EF1 + PO allocation takes between $2m$ (Theorem 30) and $O\left(m^2 \log m\right)$ (Theorem 31) comparisons; is it $O(m)$?

Search versus certificates. Is there a natural predicate $\mathcal{P}$ in the setting of Theorem 1 whose search complexity exceeds $\max\{E(\mathcal{P}), E_\emptyset(\mathcal{P})\}$ by a factor of $\omega(\log(nm))$? Remark 1 gives an artificial one, and for PO on goods the factor is $\Theta((n + m\log n)/(n+m))$ (Proposition 4, Theorem 25, and Remark 9). EF with two agents would be a natural one if $E_\emptyset(\mathrm{EF}) = o\left(m^2/\log m\right)$, as EF allocations have certificates of size 2, whereas search needs $\Omega\left(m^2\right)$ comparisons (Theorem 12); only $E_\emptyset(\mathrm{EF}) = O\left(m^2 \log m\right)$ is known (Theorem 8). For decision, is the complexity polynomial in m and the certificate sizes of the two answers, independently of n? Remark 1 rules this out for search.

Facet counts. For fixed $n \geq 3$, by Remark 7, $\frac{1}{m}\log$ of $\mathrm{DC}_{\mathrm{rand}}(\mathrm{EF})$ and of $\mathrm{SR}_{\mathrm{det}}(\mathrm{EF})$, on monotone non-degenerate additive instances, and of $f(m)$ have the same limits superior as $m \to \infty$, and the same limits inferior, all between $\log\varphi$ and $\log\vartheta$; what are they? An EF algorithm with a base below ϑ would improve the upper bound of Eq. (2), and an EF lower bound with a base above φ would refute Searles' conjecture $f(m) = F_{m+1}$. A related question is whether sorting 2^M takes $\Theta(f(m))$ comparisons (Corollary 2), which would sharpen the question of Fishburn et al. [68] on $c(m)$ (Remark 2).

A Omitted Details of Section 2

A.1 Sections 2.1 and 2.2

Proof of Lemma 1. Each answer b_h prescribes a convex subset of $\mathbb{R}^M$: the closed halfspace given by $\langle x, h\rangle \le 0$ if $b_h = 1$, and the open halfspace given by $\langle x, h\rangle > 0$ if $b_h = 0$. If $b_h = b_{-h} = 0$ for some $h \in \mathcal{H}_M$, then these two answers agree with no $x \in \mathbb{R}^M$. If $b_h = b_{-h} = 1$ for some $h \in S$, then these two answers force $\langle x, h\rangle = 0$, which D excludes. In both cases, $Q \coloneqq \{h, -h\}$ has $|Q| = 2 \le m + 1$. Otherwise, every $x \in K$ that agrees with b on $\mathcal{H}_M$ satisfies $\langle x, h\rangle \ne 0$ for every $h \in S$, hence lies in D, and has type b, which is impossible. Therefore K and the sets prescribed by the answers have an empty intersection. As they are finitely many convex subsets of $\mathbb{R}^M$, Helly's theorem [88] yields at most $m+1$ of them whose intersection is already empty; this holds without any closedness assumption. The answers among them form the desired Q. □

Proof of Lemma 2. The ternary arrangement consists of $L \coloneqq (3^m - 1)/2$ hyperplanes, one for each pair $\pm h$. Every face other than $\{\mathbf{0}\}$ spans a flat of some codimension $c \le m - 1$ that is the intersection of c of these hyperplanes, and it is a chamber of the arrangement induced on this flat by at most $L - c$ of the remaining hyperplanes. A central arrangement of H hyperplanes in $\mathbb{R}^r$ has at most $2\sum_{j=0}^{r-1}\binom{H-1}{j}$ chambers [54, 154]. Hence, using $\binom{L}{c}\binom{L-c}{j} = \binom{L}{s}\binom{s}{c}$ for $s = c + j$ and $2^s\binom{L}{s} \le (2L)^s$, the number of faces is at most

$$1 + 2\sum_{c+j\le m-1}\binom{L}{c}\binom{L-c}{j} = 1 + 2\sum_{s=0}^{m-1} 2^s\binom{L}{s} \le 1 + 2\,(2L+1)^{m-1} = 1 + 2\cdot 3^{m(m-1)}. \tag{21}$$

Since $1 + 2\cdot 3^{m(m-1)} \le 3^{m^2-m+1}$ and $\left(m^2 - m + 1\right)\log_2 3 < 1.6\,m^2$, the first claim follows. The type of an instance consists of the types of its agents, which coincide for homogeneous instances. □

The upper bound of Theorem 1 rests on the following form of Moshkov's majority-vote argument [122, Theorem 1], [123, Theorem 3.6], which allows several valid outputs. For a nonnegative integer t, let $\rho(t) \coloneqq \max_{1\le q\le t} q/\log_2 \max\{2, q\}$, with $\rho(0) \coloneqq 0$, so that ρ is non-decreasing and $\rho(t) = t/\log_2 t$ for $t \ge 4$.

Lemma 6. *Let $B \subseteq \{0,1\}^D$ be a finite nonempty set of vectors indexed by a finite set D, and let each $b \in B$ have a nonempty set $R(b)$ of valid outputs. For $h \in \{0,1\}^D$, let $c(h)$ be the least size of a set $Q \subseteq D$ such that the members of B that agree with h on Q either have a common valid output or do not exist, and let $t \coloneqq \max_{h\in\{0,1\}^D} c(h)$. Then there is a deterministic algorithm that queries the entries of an unknown $b \in B$ and outputs a member of $R(b)$ after at most $t + \rho(t)\log_2|B|$ queries.*

Proof. The algorithm maintains the set $V \subseteq B$ of the vectors consistent with the answers so far, starting from $V = B$, and proceeds in phases. In each phase, let h be a coordinatewise majority vector of V, i.e., h_c is a most frequent value of b_c over $b \in V$ for each $c \in D$. Since the members of V form a subset of B, there is a set Q of at most t coordinates such that the members of V that agree with h on Q have a common valid output or do not exist. The algorithm queries the coordinates of Q one at a time, each time choosing an unqueried $c \in Q$ that maximizes the number of current members of V disagreeing with h at c, and discards the members inconsistent with the answer. If all answers agree with h, the remaining members have a common valid output, which the algorithm returns; this final phase costs at most t queries. Otherwise, suppose that the first disagreement occurs at the qth query of the phase. The sets of members discarded by the first

$q-1$ queries are disjoint, and by the greedy choice each of them is at least as large as the set of members that the qth query keeps, namely those disagreeing with h at its coordinate; so the phase leaves at most $|V|/q$ members, and as h is a majority vector, also at most $|V|/2$ members. Hence a phase with q queries shrinks V by a factor of at least $\max\{2, q\}$, so it costs at most $\rho(t)$ queries per halving of $|V|$, and all phases but the last cost at most $\rho(t)\log_2|B|$ queries in total. □

Proof of Theorem 1. Let $B \coloneqq \{\beta(\mathcal{I}) \mid \mathcal{I} \in \mathcal{C}\}$ denote the set of all types realized by instances in $\mathcal{C}$.

For the lower bounds, consider first a deterministic algorithm Λ that is correct on $\mathcal{C}$. If Λ queries the coordinates Q on $\mathcal{I} \in \mathcal{C}$ and outputs ω, then Q is a certificate of ω at $\mathcal{I}$, or an emptiness certificate if ω reports $\mathcal{P}(\mathcal{I}) = \emptyset$: every $\mathcal{I}' \in \mathcal{C}$ whose type agrees with $\beta(\mathcal{I})$ on Q receives the same answers, so Λ outputs ω on $\mathcal{I}'$ as well, which is valid by correctness. Applying this to an instance that attains $E(\mathcal{P})$ or $E_\emptyset(\mathcal{P})$ gives $\mathrm{SR}_{\mathrm{det}}(\mathcal{P}) \geq E$. For a randomized algorithm Λ that is correct with probability 1 on every $\mathcal{I} \in \mathcal{C}$, fix one representative instance of each type in B, and let Λ_s denote the deterministic algorithm obtained by fixing the internal randomness s. Since B is finite by Lemma 2, almost every s makes Λ_s correct on all the representatives, hence on all of $\mathcal{C}$, because Λ_s behaves identically on instances of the same type. For such s, Λ_s uses at least E queries on an instance that attains E, which is fixed in advance; taking the expectation over s gives $\mathrm{SR}_{\mathrm{rand}}(\mathcal{P}) \geq E$.

For the upper bound, we apply Lemma 6 with $D \coloneqq N \times \mathcal{H}_M$, the set B, and $R(b) \coloneqq \mathcal{P}(\mathcal{I})$ for any $\mathcal{I} \in \mathcal{C}$ of type b if this set is nonempty, and $R(b) \coloneqq \{\mathrm{EMPTY}\}$ otherwise; each query of an entry is a comparison query. For $h \in B$, we have $c(h) \leq E$ by the definition of E. For $h \notin B$, since $\mathcal{C}$ is a product of the domains D_i, some agent $i \in N$ has a part $(h_{i,g})_{g \in \mathcal{H}_M}$ that is the type of no $x \in D_i$, and Lemma 1 yields at most $m+1$ coordinates of agent i on which h agrees with no member of B, so that $c(h) \leq m+1$. Therefore, $t \leq \max\{E, m+1\} \leq E'$. As ρ is non-decreasing and $E' \geq 4$, Lemmas 2 and 6 give at most $E' + \rho(E')\log_2|B| < E' + 1.6\,E'nm^2/\log_2 E'$ queries. The additive term E' is absorbed in the O-bound, because there are only $n(3^m-1)$ coordinates, so that $E' \leq n(3^m-1)$ and $\log_2 E' = O(nm^2)$.

For the homogeneous instances, the types are the vectors $(b_1, \ldots, b_1)$ with b_1 the type of some $x \in D_1$, and there are fewer than $2^{1.6m^2}$ of them by Lemma 2. If $h \notin B$ has two agents whose parts differ at some $g \in \mathcal{H}_M$, then the two corresponding coordinates agree with no member of B; otherwise all parts of h coincide with a vector that is the type of no $x \in D_1$, and Lemma 1 applies. The rest of the proof is unchanged, except that the additive term E' is absorbed because $\log_2 E' = O(m)$: since all agents share one valuation, the coordinates of agent 1 alone form a certificate whenever any set does, so $E \leq 3^m - 1$. □

Proof of Corollary 1. Let Λ be a verifier that is correct with probability 1, fix one representative instance of each type in $\{\beta(\mathcal{I}) \mid \mathcal{I} \in \mathcal{C}\}$, and let Λ_s denote Λ with its internal randomness fixed to s. There are finitely many pairs of a representative and an allocation, so almost every s makes Λ_s correct on all of them, and hence on all of $\mathcal{C}$. For such s, $\mathcal{I} \in \mathcal{C}$, and $A \in \mathcal{P}(\mathcal{I})$, every $\mathcal{I}' \in \mathcal{C}$ whose type agrees with $\beta(\mathcal{I})$ on the coordinates that Λ_s queries on $(\mathcal{I}, A)$ makes Λ_s accept A, so these coordinates form a certificate of A at $\mathcal{I}$. Taking the expectation over s shows that the least size of a certificate of A at $\mathcal{I}$ is at most $\mathrm{VR}(\mathcal{P} \mid \Lambda)$, so $E(\mathcal{P}) \leq \mathrm{VR}_{\mathrm{rand}}(\mathcal{P})$, while $E_\emptyset(\mathcal{P}) = 0$. Thus $E' \leq V'$ in Theorem 1, and the claim follows because $x \mapsto x/\log_2 x$ is increasing for $x \geq 4$. □

A.2 The Construction of Remark 1

The predicate of Remark 1 also has these properties on the monotone instances, the strictly positive ones, and the non-degenerate strictly positive ones. It asks for a violated constraint of an inconsistent parity system on a cubic expander [151], a standard example in the study of search

problems in the decision-tree model [114], whose variables are read by single comparisons. Hence no bound of the form $g(E(\mathcal{P})) \cdot o(nm)$ holds for all such predicates. Since $E' = \max\{m+1, 4\}$ here, the example forces a factor $\Omega(n)$ over E' in Theorem 1.

Let $s := \lfloor m/2 \rfloor$, and partition $2s$ of the items into pairs $\{a_j, b_j\}$ for $j \in \{0, 1, \ldots, s-1\}$. For an additive instance $\mathcal{I}$, agent $i \in N$, and $j < s$, let the *slot bit* be $x_{i,j}(\mathcal{I}) := 1 - [v_i(\{a_j\}) \leq v_i(\{b_j\})]$, which is the negation of one comparison answer and equals 0 in case of a tie. Let $r := 2\lfloor ns/3 \rfloor$, and take a connected cubic graph $G = (V, \mathcal{E})$ on r vertices with edge expansion $h_0 > 0$, i.e., $|\delta(S)| \geq h_0|S|$ for every $S \subseteq V$ with $|S| \leq r/2$. For every sufficiently large even r, random cubic graphs have edge expansion greater than $2/11$ with high probability, as shown by Bollobás; see Kolesnik and Wormald [106], whose bound is about 0.184. Fix charges $\chi \in \{0,1\}^V$ with an odd sum, an injection of $\mathcal{E}$ into the slots (i, j), which is possible as $|\mathcal{E}| = 3r/2 \leq ns$, and an injection $w \mapsto A_w$ of V into the complete allocations, which is possible as $r < ns \leq n^m$. Writing x_e for the slot bit assigned to $e \in \mathcal{E}$, a vertex w is *violated* if $\bigoplus_{e \ni w} x_e \neq \chi(w)$, and we let

$$\mathcal{P}(\mathcal{I}) := \{A_w \mid w \in V \text{ is violated}\}. \tag{22}$$

Each x_e occurs twice in $\bigoplus_{w \in V} \bigoplus_{e \ni w} x_e = 0$, whereas $\bigoplus_w \chi(w) = 1$, so some vertex is violated and $\mathcal{P}(\mathcal{I}) \neq \emptyset$. The predicate is determined by comparisons, and the three coordinates of the edges at a violated vertex w form a certificate of A_w, so $E(\mathcal{P}) \leq 3$. Reading all slot bits shows $\mathrm{SR}_{\det}(\mathcal{P}) \leq ns \leq nm/2$, and a candidate A_w is verified by the three comparisons of the edges at w, while every other allocation is rejected without queries, so $\mathrm{VR}_{\det}(\mathcal{P}) \leq 3$.

The hard family. For bits $(x_{i,j})$ with $x_{i,j} = 0$ at the unused slots, let every agent i have the values $v_i(\{a_j\}) := 4^j(1 + x_{i,j})$ and $v_i(\{b_j\}) := 4^j(2 - x_{i,j})$, and value $1/4$ for the remaining item if m is odd. These instances are strictly positive, and their slot bits are the given bits. For a ternary normal $h \in \mathcal{H}_M$ that involves only the remaining item, $\langle v_i, h \rangle = \pm 1/4$. Otherwise, let j be the largest index with $(h_{a_j}, h_{b_j}) \neq (0, 0)$. The contribution $h_{a_j} v_i(\{a_j\}) + h_{b_j} v_i(\{b_j\})$ has absolute value at least 4^j, its sign depends on $x_{i,j}$ only if $h_{a_j} = -h_{b_j}$, and the other items contribute at most $3\sum_{t<j} 4^t + 1/4 < 4^j$ in absolute value. Hence on this family every comparison query is either constant or reveals a single slot bit, and no two bundles tie, so the family is non-degenerate. In particular, two coordinates reveal at most two slot bits, so at an instance of the family with a unique violated vertex w, any two coordinates leave some edge at w unread, and flipping its bit yields an instance of the family that agrees on them but has $\mathcal{P} = \{A_{w'}\}$ for some $w' \neq w$; thus $E(\mathcal{P}) = 3$, already on the non-degenerate strictly positive instances, and $\mathrm{VR}_{\det}(\mathcal{P}) = 3$ because $E(\mathcal{P}) \leq \mathrm{VR}_{\det}(\mathcal{P})$.

The lower bound. Draw W uniformly from V, and then the edge bits uniformly among the labelings whose unique violated vertex is W; for a connected graph, every W has the same number $2^{|\mathcal{E}|-r+1}$ of them, since the image of the incidence map is the set of even vectors. After any sequence of queries, each revealing at most one edge bit, let H be G without the revealed edges, and let the residual charge of a vertex be χ plus the revealed bits at it. For each w, the unrevealed bits must solve the parity system on H whose right-hand side is the residual charge plus the indicator of w; it is solvable if and only if every component of H has an even total of this right-hand side, and then it has $2^{|E(H)|-r+\kappa(H)}$ solutions, where $\kappa(H)$ is the number of components. Hence exactly one component C of H has an odd total residual charge, the possible values of W are the vertices of C, and conditionally on the answers, W is uniform on C. If $q < h_0 r/2$ edges are revealed, then H has a unique component L with more than $r/2$ vertices, and $|V \setminus L| \leq q/h_0$: otherwise every component D of H has at most $r/2$ vertices, and summing $|\delta(D)| \geq h_0|D|$ over them gives

$h_0 r \le 2q$, since each revealed edge lies in at most two of these cuts; and, as $|V \setminus L| < r/2$ and every edge of $\delta(V \setminus L) = \delta(L)$ is revealed, the expansion of $V \setminus L$ gives $h_0 |V \setminus L| \le q$. These giant components shrink as more edges are revealed, and while $W \in L$, the next query moves W out of the giant component only if it reveals a bridge of L, in which case this happens with probability $|S'|/|L| \le 2|S'|/r$, where S' is the part split off. The parts split off are disjoint and lie outside the final giant component, so their total size is at most q/h_0, and W leaves the giant component within q queries with probability at most $2q/(h_0 r)$. A deterministic algorithm that is correct on all instances cannot stop while C has two or more vertices, since each of them is the unique valid output on some instance consistent with the answers. For $q := \lfloor h_0 r/8 \rfloor$, it therefore makes more than q queries with probability at least $3/4$, so its expected number of queries over W and the bits is at least $3q/4 = \Omega(nm)$. For a randomized algorithm that is correct with probability 1, almost every fixing of its internal randomness yields such a deterministic algorithm on the finitely many types, so some instance has expected cost $\Omega(nm)$. For an algorithm that is correct with probability at least $1 - \varepsilon$ for some $\varepsilon \in (0, 1/2)$, fix its internal randomness. If it stops within $q < h_0 r/2$ queries while W is still in the giant component, then given the answers, W is uniform on a component of more than $r/2$ vertices, so its output is correct with probability less than $2/r$. Hence it returns a correct output within q queries with probability at most $2q/(h_0 r) + 2/r$, and since it is correct with probability at least $1 - \varepsilon$, it makes more than q queries with probability at least $1 - \varepsilon - 2q/(h_0 r) - 2/r$. For $q := \lfloor (1-\varepsilon) h_0 r/4 \rfloor$, this is at least $(1-\varepsilon)/2 - 2/r$, so averaging over the internal randomness and the drawn instance shows that some instance has expected cost $\Omega\big((1-\varepsilon)^2 nm\big)$, which is $\Omega(nm)$ for fixed ε. The instances drawn here are those of the hard family, so the bounds hold on all four classes of Remark 1.

A.3 Section 2.3

Proof of Corollary 2. The order of 2^M with respect to v determines and is determined by the type of v, so the task is to identify the type of v among the set B of all types of a single agent, i.e., $R(b) := \{b\}$. For $h \notin B$, Lemma 1 with $K = \mathbb{R}^M$ and $S = \emptyset$ gives $c(h) \le m + 1$. The type of $\mathbf{0}$ is identified by the $m + 1$ comparisons $[v(\{g\}) \le v(\emptyset)]$ for $g \in M$ and $[v(\emptyset) \le v(M)]$, whose answers TRUE force $v = \mathbf{0}$. For any other $h \in B$, let F be the face of the ternary arrangement with type h, of dimension $d \ge 1$, and choose a chamber C whose closure contains F; let C have $k \le f(m)$ facets. The normals of the facets containing F span the orthogonal complement of the linear span of F, of dimension $c := m - d$. Query both orientations of c linearly independent ones among them, which forces x into the span of F, and, for each facet of C not containing F, the orientation whose answer at h is FALSE, which is a strict inequality. Every $x \in \mathbb{R}^M$ that agrees with h on these at most $2c + (k - c) \le f(m) + m - 1$ coordinates lies in the closure of C, and its tight facet inequalities are exactly those containing F. In a polyhedral cone, the points whose tight facet inequalities are exactly those of a face F form the relative interior of F, on which every comparison is constant; thus x has type h. Since every chamber is a pointed cone in $\mathbb{R}^M$ and thus has at least m facets, $c(h) \le f(m) + m - 1 \le 2f(m)$ for every h, and Lemmas 2 and 6 give at most $2f(m) + 1.6 \cdot 2f(m)m^2/\log_2(2f(m))$ queries, which is $O(mf(m))$ because $f(m) \ge F_{m+1} \ge \varphi^{m-1}$ by Eq. (2).

For the lower bound, let C be a chamber with $f(m)$ facets and $v \in C$. For each facet of C, with normal $\pm g$, Lemma 13 gives a chamber C_g across the facet whose points differ from those of C only in the comparisons with normal $\pm g$. Thus a deterministic algorithm that asks no comparison with normal $\pm g$ on v behaves identically on v and on the points of C_g, and is wrong on one of them. Hence every correct deterministic algorithm asks at least $f(m)$ comparisons on v, and the bound for randomized algorithms that are correct with probability 1 follows from the same

argument over the finitely many types as in the proof of Theorem 1. For error ε, pick one point v_g in each chamber C_g; the sets $\Delta(v, v_g)$ consist of the comparisons with normal $\pm g$ and are pairwise disjoint, and the orders of v and of the points v_g are pairwise distinct, so Lemma 12 gives the bound $(1-\varepsilon)f(m) - \varepsilon$. □

Integer values. We prove the claim after Corollary 3 that $O(nm)$ queries identify the types of all agents if W is fixed. Consider one agent, whose item values are integers in $\{-W, \dots, W\}$, and write v for its valuation function. First, $2m$ queries compare each item with $\emptyset$ and give the set M^- of items of negative value. The additive function u with $u(\{g\}) = |v(\{g\})|$ for $g \in M$ satisfies $u(X) = v(X \triangle M^-) - v(M^-)$, so comparing $u(X)$ with $u(Y)$ is comparing $v(X \triangle M^-)$ with $v(Y \triangle M^-)$. Next, binary insertion sorts the items of positive u-value into $k \le W$ classes of equal value with $O(m \log(W+1))$ queries. Then v is determined by the values of these classes, an increasing sequence in $\{1, \dots, W\}$, so at most $\binom{W}{k}$ candidates for v remain. As long as two remaining candidates differ in type, the algorithm queries a coordinate on which they differ, which discards at least one candidate and never the true one. After at most $\binom{W}{k} - 1 < \binom{W}{\lfloor W/2 \rfloor}$ such queries, all remaining candidates have the type of v. For fixed W, this is $O(m)$ queries per agent, and $O(nm)$ in total.

A.4 Section 2.4

Theorems 2 to 4 run the decision tree of Lemma 4 for sets H of the following form, at whose inner nodes comparisons suffice.

Lemma 7. *Let M be a finite set with $m := |M| \ge 2$, let $c \in \mathbb{Z}^M$ and $\lambda \in \{1, 2\}$, and let $\mathcal{T} \subseteq 2^M$ be such that $H := \{c - \lambda \mathbf{1}_{M|T} \mid T \in \mathcal{T}\} \subseteq \{-1, 0, 1\}^M$. Given a queryable additive function $v : 2^M \to \mathbb{R}$, one can compute $\ell_H(v)$ using $O(m \log m \log(|\mathcal{T}| + 1))$ comparison queries.*

Proof. As $b = 1$ and $|H| \le |\mathcal{T}|$, Lemma 4 gives a decision tree of depth $O(m \log m \log(|\mathcal{T}| + 1))$ that computes ℓ_H, and we run it on v. By additivity, each of its inner nodes costs at most two comparison queries: for $h \in H$, $\operatorname{sign}(\langle v, h \rangle) = \operatorname{sign}(v(\{g \in M \mid h_g = 1\}) - v(\{g \in M \mid h_g = -1\}))$, and, for $T_1, T_2 \in \mathcal{T}$, $\operatorname{sign}\left(\langle v, (c - \lambda \mathbf{1}_{M|T_1}) - (c - \lambda \mathbf{1}_{M|T_2}) \rangle\right) = \operatorname{sign}(v(T_2) - v(T_1))$, as $\lambda > 0$. □

Proof of Theorem 2. Let $m := |M|$. Lemma 7 with $c := \mathbf{1}_{M|S}$, $\lambda := 1$, and $\mathcal{T} := 2^M$ computes $\ell_H(v)$ for $H = \{\mathbf{1}_{M|S} - \mathbf{1}_{M|T} \mid T \subseteq M\}$ using $O(m^2 \log m)$ comparison queries, and $\ell_H(v)$ gives $\operatorname{sign}(v(T) - v(S)) = -\operatorname{sign}\left(\langle v, \mathbf{1}_{M|S} - \mathbf{1}_{M|T} \rangle\right)$ for every $T \subseteq M$. □

Proof of Theorem 3. Let $m := |M|$. Lemma 7 with $c := \mathbf{1}_{M|M}$, $\lambda := 2$, and $\mathcal{T} := 2^M$ computes $\ell_H(v)$ for the set H of the vectors $h_T := \mathbf{1}_{M|M} - 2 \cdot \mathbf{1}_{M|T} = \mathbf{1}_{M|M \setminus T} - \mathbf{1}_{M|T}$ with $T \subseteq M$ using $O(m^2 \log m)$ comparison queries, and $\ell_H(v)$ gives $\operatorname{sign}(v(T) - v(M \setminus T)) = -\operatorname{sign}(\langle v, h_T \rangle)$ for every $T \subseteq M$. □

Proof of Theorem 4. Let $m := |M|$; as $\mathcal{B}_w = \mathcal{B}_m$ for $w > m$, we may assume $w \le m$. For disjoint $A, B \subseteq M$, the sets S with $A \subseteq S \subseteq M \setminus B$ form the subcube $Q(A, B)$. The family $\mathcal{Q}$ of the subcubes $Q(A, B)$ with $|A \cup B| = w$ has $\binom{m}{w} 2^w$ members, each containing a 2^{-w} fraction of the subsets of M. We choose pivots greedily, each time a set $S \subseteq M$ that lies in the most subcubes in $\mathcal{Q}$ containing no pivot chosen so far. If U such subcubes remain, some S lies in at least $U 2^{-w}$ of them, so at most $U(1 - 2^{-w})$ remain after the choice. Let $t := \left\lfloor 2^w \ln\left(\binom{m}{w} 2^w\right) \right\rfloor + 1$, so that $t \le 2^w w \ln(2m) + 1$. After t choices, fewer than $\binom{m}{w} 2^w e^{-t 2^{-w}} \le 1$ subcubes remain, so the chosen family $\mathcal{S}$ of at most $t = O(2^w w \log m)$ pivots meets every subcube in $\mathcal{Q}$.

For each $S \in \mathcal{S}$, Lemma 7 with $c \coloneqq \mathbf{1}_{M|S}$, $\lambda \coloneqq 1$, and $\mathcal{T} \coloneqq \{T \subseteq M \mid 1 \le |S \triangle T| \le w\}$, of size at most $(m+1)^w$, learns $\operatorname{sign}(v(S) - v(T))$ for every T with $|S \triangle T| \le w$ using $O\left(wm \log^2 m\right)$ comparison queries, and so $O\left(w^2 2^w m \log^3 m\right)$ over all pivots.

Now let $X, Y \subseteq M$ with $|X \triangle Y| \le w$; if $X = Y$, the sign is 0, so suppose $X \ne Y$. The sets $A \coloneqq X \setminus Y$ and $B \coloneqq Y \setminus X$ are disjoint, $|A \cup B| \le w$, and $v(X) - v(Y) = v(A) - v(B)$. Since $w \le m$, some $A' \supseteq A$ disjoint from B satisfies $|A' \cup B| = w$, and some $S \in \mathcal{S}$ lies in $Q(A', B) \subseteq Q(A, B)$. Then $T \coloneqq (S \setminus A) \cup B$ satisfies $S \triangle T = A \cup B$, so $\operatorname{sign}(v(S) - v(T))$ is known, and it equals $\operatorname{sign}(v(X) - v(Y))$, as $v(S) - v(T) = v(A) - v(B)$ by additivity. □

A.5 Section 2.5

Sieving. Theorems 5 and 7 rest on one sieve, which we state abstractly. We use twice that an arrangement of h hyperplanes in $\mathbb{R}^k$ has at most $(2h+1)^k$ faces: choose a point in each face and replace each hyperplane by two parallel copies close enough to it that these points lie in distinct chambers of the new arrangement, which has at most $\sum_{j=0}^{k} \binom{2h}{j} \le (2h+1)^k$ chambers.

Let $\mathcal{X}$ be a finite set of candidates. Each $v \in \mathbb{R}^M$ gives each candidate x a score $q_v(x) \in \mathbb{R}$ and ranks $\mathcal{X}$ by decreasing score, with ties broken by a fixed total order on $\mathcal{X}$; for nonempty $S \subseteq \mathcal{X}$, let $b_v(S)$ denote the first member of S in this ranking. Suppose that each nonempty $S \subseteq \mathcal{X}$ and each $v \in \mathbb{R}^M$ determine a set $\mathrm{El}(S, v) \subseteq \mathcal{X}$ of *eliminated* candidates, and that for some $\kappa \in \{1, 2, \ldots\}$ and $P : \{1, 2, \ldots\} \to \{1, 2, \ldots\}$, the following hold for all such S and v:

(S1) $q_v(x) \le q_v(b_v(S))$ for every $x \in \mathrm{El}(S, v)$;

(S2) $S \subseteq \mathrm{El}(S, v)$;

(S3) $\mathrm{El}(T, v) \subseteq \mathrm{El}(S, v)$ for every nonempty $T \subseteq S$;

(S4) if $|S| \le t$, then the sets $\mathrm{El}(S, v')$ with $v' \in \mathbb{R}^M$ are at most $P(t)$ in number, and they can be computed;

(S5) if $x^1, \ldots, x^k \in \mathcal{X}$ appear in this order in the ranking and $x^t \notin \mathrm{El}\left(\{x^1, \ldots, x^{t-1}\}, v\right)$ for $2 \le t \le k$, then $k \le \kappa$.

Lemma 8. *Let $s \ge 48\kappa$ be an integer with $P(2\kappa)e^{-s/288} \le 1/8$. Then every $C \subseteq \mathcal{X}$ with $|C| \ge s$ has a subset S of size s such that $|\mathrm{El}(S, v) \cap C| \ge |C|/8$ for every $v \in \mathbb{R}^M$, and such an S can be computed without queries.*

Proof. First, let $C' \subseteq \mathcal{X}$ with $|C'| \ge 2\kappa + 1$ and $v \in \mathbb{R}^M$, and let T be a uniformly random subset of C' of size 2κ. We show that $|\mathrm{El}(T, v) \cap C'| \ge |C'|/4$ with probability at least $1/3$. Adding a uniformly random $x \in C' \setminus T$ to T gives a uniformly random subset R of C' of size $2\kappa + 1$, of which x is a uniformly random member. List R as $r_1, \ldots, r_{2\kappa+1}$ in the ranking, and let J consist of 1 and the indices $t \ge 2$ with $r_t \notin \mathrm{El}(\{r_1, \ldots, r_{t-1}\}, v)$. By (S3), the members r_t with $t \in J$ satisfy the hypothesis of (S5), so $|J| \le \kappa$. Every $t \notin J$ satisfies $r_t \in \mathrm{El}(\{r_1, \ldots, r_{t-1}\}, v) \subseteq \mathrm{El}(R \setminus \{r_t\}, v)$ by (S3), so $x \in \mathrm{El}(T, v)$ with probability at least $(\kappa + 1)/(2\kappa + 1) > 1/2$. As $T \subseteq \mathrm{El}(T, v)$ by (S2) and x is uniform in $C' \setminus T$ given T, the expectation of $|\mathrm{El}(T, v) \cap C'|$ is at least $2\kappa + (|C'| - 2\kappa)/2 \ge |C'|/2$. Since this number is at most $|C'|$, it is at least $|C'|/4$ with probability at least $(1/2 - 1/4)/(1 - 1/4) = 1/3$.

Now let S be a uniformly random subset of C of size s, and T a uniformly random subset of S of size 2κ. Then T is a uniformly random subset of C of size 2κ, and given T, the set $S \setminus T$ is a uniformly random subset of $C \setminus T$ of size $s - 2\kappa$. Call S *bad* if some v satisfies $|\mathrm{El}(S, v) \cap C| < |C|/8$,

and call (S,T) bad if some v satisfies $|\mathrm{El}(T,v)\cap C|<|C|/8$ and $|\mathrm{El}(T,v)\cap S|\geq s/4$. If S is bad with a witness v, then $|\mathrm{El}(T,v)\cap C|<|C|/8$ by (S3), so the first part, applied to $C'\coloneqq S$, shows that (S,T) is bad with probability at least $1/3$ given S. It therefore suffices to show that (S,T) is bad with probability at most $1/8$, since then S is bad with probability at most $3/8<1$. Fix T. By (S4), the sets $\mathrm{El}(T,v)$ are at most $P(2\kappa)$ in number; fix one of them, E, with $|E\cap C|<|C|/8$. As $T\subseteq E$ by (S2), $|E\cap S|\geq s/4$ requires $X\coloneqq|E\cap(S\setminus T)|\geq s/4-2\kappa$, whereas the expectation of X is at most $(s-2\kappa)\,\frac{|C|/8}{|C|-2\kappa}\leq\frac{s-2\kappa}{6}$, because $|C|\geq s\geq 8\kappa$. The gap is at least $s/12-2\kappa\geq s/24$, so Hoeffding's inequality for sampling without replacement [89] bounds the probability by $\exp\left(-2\left(s/24\right)^2/s\right)=e^{-s/288}$, and a union bound over E gives at most $P(2\kappa)e^{-s/288}\leq 1/8$. Thus some S is not bad, and since this can be checked through the finitely many sets in (S4), such an S can be found by enumeration. □

Lemma 9. *For s as in Lemma 8, a member of $\mathcal{X}$ of largest score can be found in $O\left(\log\left(|\mathcal{X}|+1\right)\right)$ rounds and a final step, where each round needs $\mathrm{El}(S,v)$ and $b_v(S)$ for one set S of size s, which is itself computed without queries, and the final step finds a member of largest score among at most s candidates.*

Proof. Start from $C\coloneqq\mathcal{X}$. While $|C|>s$, take $S\subseteq C$ as in Lemma 8 and replace C by $(C\setminus\mathrm{El}(S,v))\cup\{b_v(S)\}$; finally, return a member of C of largest score. If a member of C with the largest score in $\mathcal{X}$ is eliminated, then $b_v(S)$ also has this score by (S1), so C always contains a member of $\mathcal{X}$ of largest score. Each round leaves at most $7|C|/8+1\leq 15|C|/16$ candidates, as $|C|>s\geq 16$, so there are $O\left(\log\left(|\mathcal{X}|+1\right)\right)$ rounds. □

Both instances below bound chains by the following bound on conically independent sequences, which is the pigeonhole step in the proof of Kane et al. [101, Lemma 4.2], specialized from grids to the cube; we prove it for completeness.

Lemma 10. *Let κ_m be the largest $k\in\{1,2,\ldots\}$ with $2^{k-1}\leq(2k-1)^m$, so that $\kappa_m=O\left(m\log m\right)$. If $z_1,\ldots,z_k\in\{0,1\}^m$ satisfy $z_t-z_1\notin\mathrm{cone}\left\{z_{i+1}-z_i\mid 1\leq i\leq t-2\right\}$ for $2\leq t\leq k$, then $k\leq\kappa_m$.*

Proof. Let $d_i\coloneqq z_{i+1}-z_i\in\{-1,0,1\}^m$ for $1\leq i<k$, and suppose that $\sum_{i\in I}d_i=\sum_{i\in J}d_i$ for distinct $I,J\subseteq\{1,\ldots,k-1\}$. Let t be the largest element of $I\triangle J$, say $t\in I$. Then $d_t=\sum_{i\in J,\,i<t}d_i-\sum_{i\in I,\,i<t}d_i$, so that $z_{t+1}-z_1=\sum_{i<t}\left(1+[i\in J]-[i\in I]\right)d_i$ lies in the cone generated by $d_1,\ldots,d_{t-1}$, contrary to the hypothesis for $t+1$. Hence the 2^{k-1} sums $\sum_{i\in I}d_i$ are distinct members of $\{-(k-1),\ldots,k-1\}^m$, and $2^{k-1}\leq(2k-1)^m$. □

Proof of Theorem 5. Let $m\coloneqq|M|$, equate v with its vector in $\mathbb{R}^M$, and apply the sieve to $\mathcal{X}\coloneqq\mathcal{F}$ with the score $q_v(x)\coloneqq-v(x)$, so that the ranking is by increasing value. For nonempty $S\subseteq\mathcal{F}$, list S as $y_1,\ldots,y_r$ in this ranking, and let $\mathrm{El}(S,v)$ consist of the $x\in\mathcal{F}$ with $\mathbf{1}_{M|x}-\mathbf{1}_{M|y_1}\in\mathrm{cone}\left\{\mathbf{1}_{M|y_{i+1}}-\mathbf{1}_{M|y_i}\mid 1\leq i<r\right\}$. Each generator of this cone has a nonnegative inner product with v, so $v(x)\geq v\left(y_1\right)$ for $x\in\mathrm{El}(S,v)$, which is (S1), and $\mathbf{1}_{M|y_t}-\mathbf{1}_{M|y_1}$ is the sum of the first $t-1$ generators, which gives (S2). For (S3), the ranking of $T\subseteq S$ is a subsequence of that of S, so each generator for T and the vector $\mathbf{1}_{M|b_v(T)}-\mathbf{1}_{M|b_v(S)}$ are sums of generators for S. For (S4), $\mathrm{El}(S,v)$ depends only on the ranking of S, which is determined by the face containing v of the arrangement of the at most $|S|^2$ hyperplanes $\left\{x\in\mathbb{R}^M\mid x(y)=x(y')\right\}$ with $y,y'\in S$, where $x(y)\coloneqq\langle x,\mathbf{1}_{M|y}\rangle$; so $P(t)\coloneqq\left(2t^2+1\right)^m$ works. For (S5), Lemma 10 applies to $z_t\coloneqq\mathbf{1}_{M|x^t}$, with $\kappa\coloneqq\kappa_m$. As $\log P\left(2\kappa_m\right)=O\left(m\log m\right)$, Lemma 8 holds with some $s=O\left(m\log m\right)$. In each round of Lemma 9, the ranking of S is one of at most $P(s)$ known total orders, so the sorting bound of Fredman [71, Theorem 3.1] finds it with $\log_2 P(s)+2s=O\left(m\log m\right)$ comparisons of members of S,

each of which costs at most two queries; the ranking determines $\mathrm{El}(S, v)$ and $b_v(S)$. The final step costs $O(s)$ queries, which also compare members of $\mathcal{F}$, so the total is $O\left(m \log m \log\left(|\mathcal{F}| + 1\right)\right)$. □

Proof of Theorem 6. Let $m := |M|$. For $S \subseteq M$, *classifying* S means computing the sign vector of Theorem 2 for the pivot S, which takes $O\left(m^2 \log m\right)$ queries.

For a nonempty $\mathcal{C} \subseteq 2^M$ of size N, a multiset R of members of $\mathcal{C}$ is *balanced* if, for all $a \in \mathbb{R}^M$ and $\theta \in \mathbb{R}$, the fraction of R (counted with multiplicity) in $H_{a,\theta} := \left\{T \in \mathcal{C} \mid \langle a, \mathbf{1}_{M|T}\rangle < \theta\right\}$ differs from $\left|H_{a,\theta}\right|/N$ by at most $1/8$. The set $H_{a,\theta}$ is determined by the face containing (a, θ) of the arrangement of the N hyperplanes $\left\{(a, \theta) \in \mathbb{R}^M \times \mathbb{R} \mid \langle a, \mathbf{1}_{M|T}\rangle = \theta\right\}$ with $T \in \mathcal{C}$, so there are at most $(2N+1)^{m+1}$ such sets. For r independent uniform draws from $\mathcal{C}$, Hoeffding's inequality bounds the probability that the fraction in a fixed set deviates by more than $1/8$ by $2e^{-r/32}$, so the draws are balanced with positive probability for $r := \left\lfloor 32 \ln\left(2(2N+1)^{m+1}\right)\right\rfloor + 1 = O\left(m \log\left(N+1\right)\right)$, and a balanced R of this size can be found by enumeration without queries. If S is a median of R with respect to v, then at most half of R lies in $\{T \in \mathcal{C} \mid v(T) < v(S)\} = H_{v,v(S)}$ and at most half in $\{T \in \mathcal{C} \mid v(T) > v(S)\} = H_{-v,-v(S)}$, so each of these sets has at most $5N/8$ members.

The algorithm maintains a family $\mathcal{C} \subseteq 2^M$ that contains a set of value $\tau(v)$, starting from $\mathcal{C} := 2^M$. In each round, it computes a balanced R for $\mathcal{C}$, sorts R to find a median S, and classifies S, which determines the sign of $\tau(v) - v(S)$. If the sign is 0, it returns $S^* := S$, whose classification is the sign vector $\left(\operatorname{sign}\left(v(T) - \tau(v)\right)\right)_{T \subseteq M}$. Otherwise, it replaces $\mathcal{C}$ by $\{T \in \mathcal{C} \mid v(T) < v(S)\}$ if the sign is negative and by $\{T \in \mathcal{C} \mid v(T) > v(S)\}$ if it is positive, which keeps every member of $\mathcal{C}$ of value $\tau(v)$ and at most $5|\mathcal{C}|/8$ members in all. Since $|\mathcal{C}| \leq 2^m$ at the start, at most $O(m)$ rounds replace $\mathcal{C}$; once $\mathcal{C}$ consists of a single set, that set has value $\tau(v)$, so the next round classifies it and returns it. Each round uses $O\left(r \log r + m^2 \log m\right) = O\left(m^2 \log m\right)$ queries, as $r = O\left(m^2\right)$. □

Proof of Theorem 7. Let $m := |M|$, let $f(B) := \min_j v\left(B_j\right)$ for $B \in \Pi_d(M)$, and apply the sieve to $\mathcal{X} := \Pi_d(M)$ with the score $q_v := f$. For each B, let $\underline{B}$ be the part B_j of least value with the least index j among ties, so that $f(B) = v\left(\underline{B}\right)$. It is found with at most $\min\{d, m+1\} - 1$ comparisons of parts, each of which costs at most two queries, since B has at most m nonempty parts and its empty parts coincide. For nonempty $S \subseteq \Pi_d(M)$, list S as $A^1, \ldots, A^r$ in this ranking, let K_S be the cone generated by the vectors $\mathbf{1}_{M|\underline{A^i}} - \mathbf{1}_{M|\underline{A^{i+1}}}$ for $1 \leq i < r$, and let $\mathrm{El}(S, v)$ consist of the $B \in \Pi_d(M)$ such that $\mathbf{1}_{M|\underline{A^1}} - \mathbf{1}_{M|B_j} \in K_S$ for some j.

Each generator of K_S has a nonnegative inner product with v, so $B \in \mathrm{El}(S, v)$ implies $f(B) \leq v\left(B_j\right) \leq v\left(\underline{A^1}\right) = f\left(A^1\right)$, which is (S1). (S2) holds with $B_j := \underline{A^i}$, as $\mathbf{1}_{M|\underline{A^1}} - \mathbf{1}_{M|\underline{A^i}}$ is a sum of generators. For (S3), as in the proof of Theorem 5, $T \subseteq S$ implies $K_T \subseteq K_S$ and $\mathbf{1}_{M|\underline{b_v(S)}} - \mathbf{1}_{M|\underline{b_v(T)}} \in K_S$. For (S4), $\mathrm{El}(S, v)$ depends only on the ranking of S and the parts $\underline{A}$ for $A \in S$, which are determined by the comparisons among the at most $|S|\left(m+1\right)$ distinct parts of members of S; so $P(t) := \left(2t^2\left(m+1\right)^2 + 1\right)^m$ works. For (S5), the part $\underline{A^t}$ of A^t shows that $\mathbf{1}_{M|\underline{A^1}} - \mathbf{1}_{M|\underline{A^t}} \notin K_{\{A^1, \ldots, A^{t-1}\}}$, so Lemma 10 applies to $z_t := \mathbf{1}_{M|M\setminus\underline{A^t}}$, with $\kappa := \kappa_m$.

As $\log P\left(2\kappa_m\right) = O\left(m \log m\right)$, Lemma 8 holds with some $s = O\left(m \log m\right)$, independently of d. Each round of Lemma 9 first finds the parts $\underline{A}$ for $A \in S$ with $O\left(s \min\{d, m\}\right)$ queries. The ranking of S is then determined by the face containing v of the arrangement of the at most s^2 hyperplanes $\left\{x \in \mathbb{R}^M \mid x\left(\underline{A}\right) = x\left(\underline{A'}\right)\right\}$ with $A, A' \in S$, so it is one of at most $\left(2s^2+1\right)^m$ known total orders, and the sorting bound of Fredman [71, Theorem 3.1] finds it with $O\left(m \log s + s\right) = O\left(m \log m\right)$ comparisons, each of which costs at most two queries; this determines K_S, $\mathrm{El}(S, v)$, and $b_v(S)$. If $d > m+1$, every $B \in \Pi_d(M)$ has at least $d - m \geq 2$ empty parts, and removing $d - m - 1$ of them gives a member of $\Pi_{m+1}(M)$ with the same least part value, and conversely, so $\mu_v(d) = \mu_v(m+1)$; running the algorithm for $m+1$ parts and adding empty parts to its output, we may thus assume $d \leq m+1$. As $|\Pi_d(M)| = d^m$, there are $O\left(m \log d\right)$ rounds, and the final

step costs $O(s \min\{d, m\})$ queries. The resulting B satisfies $\min_j v(B_j) = \mu_v(d)$, and Theorem 2 with the pivot $\underline{B}$ gives the sign vector $(\operatorname{sign}(v(T) - \mu_v(d)))_{T \subseteq M}$ with $O\left(m^2 \log m\right)$ queries. The total is $O(m \log d \cdot m \log m \min\{d, m\}) + O\left(m^2 \log m\right) = O\left(m^2 \min\{d, m\} \log \min\{d, m\} \log m\right)$, as $2 \le d \le m + 1$ and $m \ge 2$. □

A.6 Section 2.6

Unless stated otherwise, the following lemmas hold for every set $\mathcal{C}$ of instances, both for algorithms that make only comparison queries and for algorithms that may also make value queries; for the latter, the complexities of Definition 3 count queries of both kinds. Fix a predicate $\mathcal{P}$ on $\mathcal{C}$ and a task of Definition 3. Its *inputs* are the instances in $\mathcal{C}$, each with an allocation for verification, and each input has a set of *valid outputs*: for search, $\mathcal{P}(\mathcal{I})$ if this set is nonempty and {EMPTY} otherwise, and for decision and verification, the correct answer. The algorithm receives the agents, the items, and, for verification, the allocation without queries, so only inputs that share these data, which we call *compatible*, can be confused; for compatible inputs $\mathcal{I}$ and $\mathcal{I}'$, let $\Delta(\mathcal{I}, \mathcal{I}')$ denote the set of queries that they answer differently. Extending Definition 4, a set Q of queries *certifies* $A \in \mathcal{P}(\mathcal{I})$ if $A \in \mathcal{P}(\mathcal{I}')$ for every $\mathcal{I}' \in \mathcal{C}$ with the same agents and items as $\mathcal{I}$ and $Q \cap \Delta(\mathcal{I}, \mathcal{I}') = \emptyset$. Throughout, for a randomized algorithm Λ, let Λ_s denote the deterministic algorithm obtained by fixing its internal randomness to s.

First, lower bounds against deterministic algorithms that are correct on a countable family, and those that count outputs, hold for zero-error randomized algorithms.

Lemma 11. *Let $\mathcal{C}'$ be a countable set of inputs and $\mathcal{I}_0 \in \mathcal{C}'$. If every deterministic algorithm that performs the task correctly on all of $\mathcal{C}'$ makes at least L queries on $\mathcal{I}_0$, then* $\operatorname{Task}_{\text{rand}}(\mathcal{P}) \ge L$. *If only comparison queries are allowed and every deterministic algorithm that performs the task correctly on all of a countable set of compatible inputs ends at distinct leaves on K of them, as when these K inputs have pairwise disjoint sets of valid outputs, then* $\operatorname{Task}_{\text{rand}}(\mathcal{P}) \ge \log_2 K$.

The lemma and its proof apply to every task whose algorithms return an output that must be valid for the input, such as sorting 2^M or computing the sign vectors of Theorems 2 and 3, with the least expected number of queries of a zero-error algorithm in place of $\operatorname{Task}_{\text{rand}}(\mathcal{P})$.

Proof. Let Λ perform the task correctly with probability 1 on every input. For each input in $\mathcal{C}'$, the seeds s for which Λ_s is wrong on it form a null set, and so does their union over the countable set $\mathcal{C}'$. Hence almost every Λ_s is correct on all of $\mathcal{C}'$ and thus makes at least L queries on $\mathcal{I}_0$, and taking the expectation over s gives the first claim. For the second, the inputs are compatible, so each Λ_s runs them through one binary decision tree, and almost every Λ_s is correct on the countable set, so the K inputs end at distinct leaves; this holds in particular if their sets of valid outputs are pairwise disjoint, since a common leaf would give one output valid for two of them. By Kraft's inequality [55], these leaves have average depth at least $\log_2 K$; averaging over s as well, some input has expected cost at least $\log_2 K$. □

With bounded error, alternatives that differ from a baseline input in disjoint sets of queries still give lower bounds, whereas a large certificate alone does not.

Lemma 12. *Let $\varepsilon \in [0, 1/2)$, let Λ be an algorithm that performs the task correctly with probability at least $1 - \varepsilon$ on every input, and let Q be the random set of queries that Λ makes on an input $\mathcal{I}_0$. Then every input $\mathcal{I}$ compatible with $\mathcal{I}_0$ satisfies*

$$\Pr[Q \cap \Delta(\mathcal{I}_0, \mathcal{I}) \neq \emptyset] + \Pr[\textit{the output of } \Lambda \textit{ on } \mathcal{I}_0 \textit{ is valid for } \mathcal{I}] \ge 1 - \varepsilon. \tag{23}$$

Hence, if inputs $\mathcal{I}_1, \dots, \mathcal{I}_K$ compatible with $\mathcal{I}_0$ have weights $\lambda_k \geq 0$ such that each query lies in $\Delta(\mathcal{I}_0, \mathcal{I}_k)$ only for indices k of total weight at most 1, for example unit weights when these sets are pairwise disjoint, and $W \coloneqq \sum_k \lambda_k$, then Λ makes at least $(1-2\varepsilon)W$ queries on $\mathcal{I}_0$ in expectation if no output valid for $\mathcal{I}_0$ is valid for any $\mathcal{I}_k$, and at least $(1-\varepsilon)W - a$ if each output is valid only for inputs $\mathcal{I}_k$ of total weight at most a; with unit weights, it makes at least $(1-\varepsilon)K - \varepsilon$ if $\mathcal{I}_0, \dots, \mathcal{I}_K$ have pairwise disjoint sets of valid outputs. If only comparison queries are allowed and compatible inputs $\mathcal{I}_1, \dots, \mathcal{I}_K$ have pairwise disjoint sets of valid outputs, then Λ makes at least $(1-\varepsilon)\log_2 K - 1$ queries in expectation on one of them.

Proof. Run Λ on $\mathcal{I}_0$ and on $\mathcal{I}$ with the same internal randomness. If the run on $\mathcal{I}_0$ makes no query in $\Delta(\mathcal{I}_0, \mathcal{I})$, then both runs, on compatible inputs, receive the same answers and thus have identical executions, with the same output if they halt. So whenever the run on $\mathcal{I}$ is correct, which happens with probability at least $1-\varepsilon$, the run on $\mathcal{I}_0$ makes a query in $\Delta(\mathcal{I}_0, \mathcal{I})$ or returns an output valid for $\mathcal{I}$, which proves Eq. (23). With $\Delta_k \coloneqq \Delta(\mathcal{I}_0, \mathcal{I}_k)$, the weight condition gives $\sum_k \lambda_k [Q \cap \Delta_k \neq \emptyset] \leq \sum_{q \in Q} \sum_{k : q \in \Delta_k} \lambda_k \leq |Q|$, so multiplying Eq. (23) by λ_k and summing over k shows that the expectation of $|Q|$ is at least $(1-\varepsilon)W$ minus the expected total weight of the k for which the output on $\mathcal{I}_0$ is valid for $\mathcal{I}_k$. This weight is at most a; if no output valid for $\mathcal{I}_0$ is valid for any $\mathcal{I}_k$, it is positive only when the output is invalid for $\mathcal{I}_0$, so its expectation is at most εW. With unit weights and pairwise disjoint sets of valid outputs for $\mathcal{I}_0, \dots, \mathcal{I}_K$, it is at most 1 and again positive only when the output is invalid for $\mathcal{I}_0$, so its expectation is at most ε.

For the last claim, draw k uniformly from $\{1, \dots, K\}$, independently of s, and let T be the transcript of the run on $\mathcal{I}_k$. A valid output determines k, so the output identifies k with error probability at most ε, and Fano's inequality [55] bounds the conditional entropy of k given s and T by $1 + \varepsilon \log_2 K$. For fixed s, the transcripts are the leaves of a binary tree, so the entropy of T given s is at most its expected length. Hence the expected number of queries, averaged over k, is at least $\log_2 K - 1 - \varepsilon \log_2 K$, and some k attains this. □

With pairwise disjoint sets and unit weights, the first form is the block-sensitivity bound of Nisan [126, Lemma 4.2] for randomized decision trees, and the last form follows from Fano's inequality [55]. A majority vote over independent runs, executed in an interleaved way since a run may not halt, achieves the amplification for decision and verification claimed in Section 2.6; for search, it fails: if three compatible inputs have the valid outputs $\{a, b\}$, $\{b, c\}$, and $\{a, c\}$, then a uniformly random output is wrong with probability $1/3$ without any query, but every algorithm that makes no query is wrong with probability at least $1/3$ on one of them.

Two kinds of alternatives with disjoint sets Δ recur. The first moves one agent's valuation across, or onto, a facet of its chamber; distinct facets change disjoint sets of comparisons by the following lemma, which also gives the lower bound of Corollary 2. We call the instances on a facet *walls*; a lower bound can use these points only if they lie in its set of instances, which, for example, a nondegeneracy promise rules out for walls.

Lemma 13. *Let C be a chamber of the ternary arrangement and F a facet of C that spans the hyperplane with normal $\pm g \in \mathcal{H}_M$. Then some point of the relative interior of F and some point outside the closure of C have types that differ from that of the points of C only at the normals g and $-g$.*

Proof. Let $H \coloneqq \{x \in \mathbb{R}^M \mid \langle x, g \rangle = 0\}$. The relative interior of F is open in H, and every other hyperplane of the arrangement meets H in a set of lower dimension, so some x in the relative interior of F lies on no other hyperplane, and so does every point of some open ball B around x. For each normal $h \neq \pm g$, the sign of $\langle \cdot, h \rangle$ is thus constant and nonzero on B; it is its sign on C,

because x lies in the closure of C. Hence x, and the points $x' \in B$ at which $\langle x', g\rangle$ has the sign opposite to that on C, have the claimed types. □

Similarly, for valuations that are not additive, a *needle* raises the values of some pairs of an agent and a bundle above a baseline, where distinct needles change disjoint sets of pairs; the same works if every needle lowers its values instead. A comparison of $v_i(X)$ with $v_i(Y)$ that holds at the baseline changes under raises only if (i, X) is raised, and one that fails only if (i, Y) is, and symmetrically under lowerings, so each comparison or value query separates at most one needle from the baseline. The second kind of alternative flips one hidden sign of a valuation whose hidden values have super-increasing magnitudes.

Lemma 14. *Let $c \in \mathbb{Z}^M$, let $U \subseteq M$ be nonempty with $\sigma \in \{-1, 1\}^U$ and an injective map $r : U \to \{0, 1, \ldots\}$, and let $x \coloneqq c + \delta \sum_{g\in U} \sigma_g 2^{r(g)} \mathbf{1}_{M|\{g\}}$ with $0 < \delta < 2^{-1-\max_{g\in U} r(g)}$. For $h \in \mathcal{H}_M$ with $\langle c, h\rangle = 0$ and $h_g \neq 0$ for some $g \in U$, let $t(h)$ be the $g \in U$ of largest rank with $h_g \neq 0$. Then the sign of $\langle x, h\rangle$ is that of $\langle c, h\rangle$ if $\langle c, h\rangle \neq 0$, it is $\sigma_{t(h)} h_{t(h)}$ if $t(h)$ is defined, and it is 0 otherwise. Hence replacing σ_g for one $g \in U$ by $-\sigma_g$ or by 0 changes the sign only at normals h with $t(h) = g$.*

Proof. Let $R \coloneqq \max_{g\in U} r(g)$, and write $\langle x, h\rangle = \langle c, h\rangle + \delta \sum_{g\in U} h_g \sigma_g 2^{r(g)}$. The second term has absolute value less than $\delta 2^{R+1} < 1$, while $\langle c, h\rangle$ is an integer, so the sign is that of $\langle c, h\rangle$ if $\langle c, h\rangle \neq 0$. Otherwise, if $t(h)$ is defined, its term has absolute value $\delta 2^{r(t(h))} > \delta \sum_{j<r(t(h))} 2^j$, which bounds the sum of the other nonzero terms, so it determines the sign; if $t(h)$ is undefined, then $\langle x, h\rangle = 0$. As only the term of $t(h)$ matters, changing σ_g to $-\sigma_g$ or to 0 leaves the sign unchanged unless $t(h) = g$. □

With one such valuation for each agent, alternatives that each change the hidden sign at one position (i, g), at distinct positions, have pairwise disjoint sets Δ, so Lemma 12 applies; this needs that only comparison queries are allowed, since a value query reveals every hidden sign in its bundle.

Finally, two lemmas bound verification by midpoints and transfer bounds to smaller predicates.

Lemma 15. *Let A be an allocation and $\mathcal{F} \subseteq \mathcal{C}$ a nonempty finite set of instances with the same agents and items, and suppose that $\mathcal{C}$ contains the instance $\frac{1}{2}(\mathcal{I} + \mathcal{J})$ whose valuation functions are the pointwise averages of those of $\mathcal{I}, \mathcal{J} \in \mathcal{F}$. If $A \in \mathcal{P}(\mathcal{I})$ for every $\mathcal{I} \in \mathcal{F}$ but $A \notin \mathcal{P}\left(\frac{1}{2}(\mathcal{I} + \mathcal{J})\right)$ for all distinct $\mathcal{I}, \mathcal{J} \in \mathcal{F}$, then $\mathrm{VR}_{\mathrm{rand}}(\mathcal{P}) \geq \log_2 |\mathcal{F}|$ if only comparison queries are allowed.*

Proof. A comparison query asks whether $v_i(X) \leq v_i(Y)$, and the instances answering TRUE, as well as those answering FALSE, form a convex set of valuation tables, so if two instances answer it alike, so does their midpoint. Hence, if a deterministic verifier that is correct on $\mathcal{F}$ and on the midpoints followed the same accepting run on two distinct $\mathcal{I}, \mathcal{J} \in \mathcal{F}$, it would follow it on $\frac{1}{2}(\mathcal{I} + \mathcal{J})$ and accept A there, which is wrong. So the members of $\mathcal{F}$ end at distinct leaves of its binary decision tree, of average depth at least $\log_2 |\mathcal{F}|$ by Kraft's inequality. For a randomized verifier that is correct with probability 1, almost every Λ_s is correct on the finite set of $\mathcal{F}$ and its midpoints; average over s. □

Lemma 16. *Let $\mathcal{P}'$ be a predicate on $\mathcal{C}$ with $\mathcal{P}'(\mathcal{I}) \subseteq \mathcal{P}(\mathcal{I})$ for every $\mathcal{I} \in \mathcal{C}$. If $\mathcal{P}'(\mathcal{I}) \neq \emptyset$ and no set of fewer than L queries certifies $A \in \mathcal{P}(\mathcal{I})$ for any $A \in \mathcal{P}'(\mathcal{I})$, then $\mathrm{SR}_{\mathrm{rand}}(\mathcal{P}') \geq L$. If $\mathcal{I}_1, \ldots, \mathcal{I}_K \in \mathcal{C}$ have the same agents and items, pairwise disjoint sets $\mathcal{P}(\mathcal{I}_k)$, and nonempty sets $\mathcal{P}'(\mathcal{I}_k)$, then the counting bounds of Lemmas 11 and 12 hold for the search of $\mathcal{P}'$.*

Algorithm 3 An algorithm for EF verification.

Input: The sets of agents N and items M of an unknown instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$ and an allocation $A \in \mathcal{A}(\mathcal{I})$.

Output: Whether $A \in \mathrm{EF}(\mathcal{I})$ or not.

```
if A ∉ A_0(I) then
    return False.
for all i ∈ N do
    for all S ∈ {A_j | j ∈ N} \ {A_i} do
        if a comparison query verifies that v_i(S) > v_i(A_i) then
            return False.
return True.
```

Proof. For the first claim, let $\mathcal{C}'$ consist of $\mathcal{I}$ and, for every set Q of fewer than L queries and every $A \in \mathcal{P}'(\mathcal{I})$, one instance $\mathcal{I}_{Q,A} \in \mathcal{C}$ with the same agents and items as $\mathcal{I}$, $Q \cap \Delta(\mathcal{I}, \mathcal{I}_{Q,A}) = \emptyset$, and $A \notin \mathcal{P}(\mathcal{I}_{Q,A})$. This set is finite, since there are finitely many queries and allocations. A deterministic algorithm that performs the search of $\mathcal{P}'$ correctly on $\mathcal{C}'$ returns some $A \in \mathcal{P}'(\mathcal{I})$ on $\mathcal{I}$. If it made a set Q of fewer than L queries there, it would run identically on $\mathcal{I}_{Q,A}$ and return $A \notin \mathcal{P}(\mathcal{I}_{Q,A}) \supseteq \mathcal{P}'(\mathcal{I}_{Q,A})$, which is wrong. So it makes at least L queries on $\mathcal{I}$, and Lemma 11 gives the claim. For the second, the sets $\mathcal{P}'(\mathcal{I}_k)$ are nonempty and pairwise disjoint, so they are the sets of valid outputs of the search of $\mathcal{P}'$ at $\mathcal{I}_k$, and Lemmas 11 and 12 apply. □

With $\mathcal{P}' = \mathcal{P}$, the first part gives the bound $E(\mathcal{P}) \leq \mathrm{SR}_{\mathrm{rand}}(\mathcal{P})$ of Theorem 1 for every set of instances, and its analogue for value queries, with certificates made of value queries. For an intersection $\mathcal{P}' = \mathcal{P} \cap \mathcal{P}''$, only the allocations in $(\mathcal{P} \cap \mathcal{P}'')(\mathcal{I})$ need large certificates, so a certificate bound that holds only for some allocations, such as those whose bundles have equal sizes, still transfers.

B Omitted Details of Section 3

Because each property in Section 3 maps every instance to the same set of allocations regardless of the set of instances on which it is considered, every upper bound there on VR, DC, or SR for $\mathcal{C}$ also holds for any subset of $\mathcal{C}$.

B.1 Section 3.1

Proof of Proposition 1. Given any instance $\mathcal{I}$ and allocation A, Algorithm 3 correctly verifies whether $A \in \mathrm{EF}(\mathcal{I})$. It rejects an incomplete A without queries, and if A is complete with k nonempty bundles, each agent compares her bundle with the $k - 1 + [k < n]$ other distinct bundles if it is nonempty, and with the k nonempty bundles otherwise. This makes $n(n-1)$ queries if $k = n$ and nk queries if $k < n$, so at most $n \min\{n-1, m\}$, as $k \leq \min\{n, m\}$. For monotone additive instances, an agent with a nonempty bundle need not compare it with $\emptyset$, and an agent with an empty bundle needs only the comparison of M with $\emptyset$, since $v_i(A_j) \leq v_i(M)$ for every j and $v_i(M) = \sum_j v_i(A_j)$; this makes $k(k-1) + n - k \leq s(s-1) + n - s$ queries. □

Proof of Theorem 8. First, suppose that $\mathcal{C}$ consists of additive instances with two agents, and let $\mathcal{I} \in \mathcal{C}$ be the unknown instance, where $\mathcal{I} = (\{1,2\}, M, (v_1, v_2))$. Theorem 3 allows us to enumerate $\mathcal{F}_1 \coloneqq \{T \subseteq M \mid v_1(T) \geq v_1(M \setminus T)\}$ and $\mathcal{F}_2 \coloneqq \{T \subseteq M \mid v_2(T) \geq v_2(M \setminus T)\}$ using

$O\left(|M|^2 \log |M|\right)$ queries. If there exists $S \in \mathcal{F}_1$ such that $M \setminus S \in \mathcal{F}_2$, then $(S, M \setminus S) \in \mathrm{EF}(\mathcal{I})$; otherwise, $\mathrm{EF}(\mathcal{I}) = \emptyset$.

Next, suppose that $\mathcal{C}$ consists of homogeneous additive instances with even numbers of agents, and let $\mathcal{I} \in \mathcal{C}$ be the unknown instance, where $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$. Theorem 3 enables us to enumerate $\mathcal{F} \coloneqq \{T \subseteq M \mid v_1(T) < v_1(M \setminus T)\}$ using $O\left(|M|^2 \log |M|\right)$ queries. If there exists $A \in \Pi_{|N|}(M)$ such that $\bigcup_{i \in N'} A_i \notin \mathcal{F}$ for any $N' \in \binom{N}{|N|/2}$, then $A \in \mathrm{EF}(\mathcal{I})$; otherwise, $\mathrm{EF}(\mathcal{I}) = \emptyset$. Indeed, for such an A, every union of $|N|/2$ bundles and its complement, which is another such union, are worth at least $v_1(M)/2$ each, so every such union is worth exactly $v_1(M)/2$, and exchanging one of its bundles for one outside it shows that all bundles have the same value; conversely, the bundles of an EF allocation have the same value.

Finally, let $\mathcal{C}$ be the set of homogeneous additive instances with n agents and m items, and let v be the common valuation function of the unknown instance $\mathcal{I}$, so that $A \in \mathcal{A}_0(\mathcal{I})$ is EF if and only if $v(A_1) = \cdots = v(A_n)$, in which case every bundle has the value $v(M)/n$. If $n > m$, every complete allocation has an empty bundle, so an EF allocation exists if and only if $v(M) = v(\emptyset)$, in which case $(M, \emptyset, \ldots, \emptyset)$ is one; two comparisons of M with $\emptyset$ decide this.

Let $2 \leq n \leq m$. If an EF allocation exists, then $\mu_v(n) = v(M)/n$, since no n-partition has all its parts above the average; and then every $B \in \Pi_n(M)$ with $\min_j v(B_j) = \mu_v(n)$ has all its parts at least $v(M)/n$, and hence equal to $v(M)/n$, so B is EF. For the first bound, Theorem 7 with $d \coloneqq n$ finds such a B with $O\left(nm^2 \log n \log m\right)$ queries, and $2(n-1)$ comparisons of B_1 with the other parts tell whether B is EF; if it is not, no EF allocation exists.

For the second bound, let $\mathcal{C}_0 \coloneqq \binom{M}{\leq \lfloor m/n \rfloor}$; every n-partition of M has a part in $\mathcal{C}_0$. The threshold $\tau(v) \coloneqq \mu_v(n)$ satisfies the hypothesis of Theorem 6, as the sign of $\mu_v(n) - v(S)$ is determined by whether some $B \in \Pi_n(M)$ has all its parts at least, or greater than, $v(S)$. We run the algorithm in the proof of Theorem 6 from $\mathcal{C} \coloneqq \mathcal{C}_0$ instead of 2^M. If an EF allocation exists, then its part in $\mathcal{C}_0$ has the value $\mu_v(n)$, and the algorithm keeps every member of $\mathcal{C}$ of value $\tau(v)$, so it stops at a pivot S with $v(S) = \mu_v(n)$; if no EF allocation exists, it either stops at such a pivot or empties $\mathcal{C}$, in which case we report that none exists. At a pivot S with $v(S) = \mu_v(n)$, the classification of S tells, for every $T \subseteq M$, whether $v(T) = v(S)$, so we can check offline whether some $B \in \Pi_n(M)$ has all its parts of value $v(S)$; such a B is EF, and if there is none, no EF allocation exists. As in the proof of Theorem 6, each round uses $O\left(m^2 \log m\right)$ queries, and each round that does not stop keeps at most $5/8$ of $\mathcal{C}$, so there are $O\left(\log\left(|\mathcal{C}_0| + 1\right)\right)$ rounds. With $k \coloneqq \lfloor m/n \rfloor \geq 1$, we have $\log\left(|\mathcal{C}_0| + 1\right) = O\left(k \log\left(em/k\right)\right) = O\left((m/n) \log\left(n+1\right)\right)$, since $m/k < 2n$. The total is $O\left(\left(m^3/n\right) \log m \log\left(n+1\right)\right)$.

The first of these two bounds is at most $m^{2.5} \log^2 m$ times a constant if $n \leq \sqrt{m}$, and so is the second if $n > \sqrt{m}$. □

Proof of Corollary 4. For additive instances, Corollary 2 sorts 2^M with respect to each v_i with $O\left(mf(m)\right)$ comparison queries. The orders determine the type of the instance, and EF is determined by comparisons, so an EF allocation, or the fact that none exists, is found offline. For monotone additive instances, one comparison of M with $\emptyset$ per agent finds the set N_0 of agents with $v_i(M) = 0$, for whom every bundle has the value 0. Every other agent i has $\max_j v_i(A_j) > 0$ for every complete allocation A, so $A_i \neq \emptyset$ if A is EF; hence no EF allocation exists if $|N \setminus N_0| > m$, and otherwise sorting 2^M for the agents in $N \setminus N_0$ suffices.

For the second form of the first bound, $f(m) \leq B_m$ by Eq. (2). Since $\lambda < 1/3$, the terms of both sums in Eq. (2) increase geometrically for j up to about $m/3$, so $2^{s_m}\binom{m}{s_m} = \Theta\left(2^m\right)$ and $B_m = \Theta\left(\binom{m}{s_m}\right) = \Theta\left(2^{m - s_m}\right)$. By Stirling's formula, $\log_2\left(2^s \binom{m}{s}\right) = s + mH(s/m) - \frac{1}{2}\log_2 m + O(1)$ for $s = \lambda m + O\left(\log m\right)$, and as $\lambda + H(\lambda) = 1$ and $H'(\lambda) = \log_2\left((1-\lambda)/\lambda\right)$, this gives $s_m =$

$\lambda m + \eta \log_2 m + O(1)$. Hence $B_m = \Theta\left(2^{(1-\lambda)m} m^{-\eta}\right) = \Theta\left(\vartheta^m m^{-\eta}\right)$, as $\vartheta = 2^{H(\lambda)} = 2^{1-\lambda}$. □

Proof of Theorem 9. For the first part, let A be an EF allocation; for each agent i, the values $v_i(A_j)$ sum to $v_i(M) > 0$, so $v_i(A_i) = \max_j v_i(A_j) > 0$ and $A_i \neq \emptyset$. Hence no EF allocation exists if $m < n$. Let $2 \leq n \leq m$ and $w \coloneqq r + 2 \leq m$. In a complete allocation with nonempty bundles, any two bundles A_i and A_j together contain at most $m - (n-2) = w$ items, so whether it is EF depends only on the signs of $v_i(X) - v_i(Y)$ with $|X \triangle Y| \leq w$. Theorem 4 computes these signs for each agent with $O\left(w^2 2^w m \log^3 m\right)$ queries, and an EF allocation, all of whose bundles are nonempty, or the fact that none exists, is then found offline.

For the second part, let v be the common valuation, with $n \leq m < 2n$. Two comparisons of M with $\emptyset$ tell whether $v(M) = 0$, in which case all item values are 0 and every complete allocation is EF, and otherwise the sign of $v(M)$, which is the sign of every nonzero item value; replacing v by $-v$ if it is negative, which exchanges the sides of each comparison and preserves EF, we may assume that the item values are nonnegative. The m comparisons of single items with $\emptyset$ then find the items of value 0, which can be added to any bundle of an EF allocation and removed from one, as all its bundles have the positive value $v(M)/n$ and thus keep a positive item. So it suffices to treat the p items of positive value, where no EF allocation exists if $p < n$, and otherwise $n \leq p \leq m < 2n$; we write m and r for p and $p - n$ below. In an EF allocation A, all bundles have the positive value $\theta \coloneqq v(M)/n$, so they are nonempty, and at least $n - r$ of them are single items: if x bundles are, the other $n - x$ have at least two items each, so $m \geq x + 2(n - x)$. So θ is the value of an item; as every item lies in a bundle of value θ, its value is at most θ, with equality only if it forms its bundle alone. Hence θ is the largest item value t, and every item of value t forms a bundle by itself. With $O(m)$ comparisons, we find an item g of value t and the set K of the k items of value t. If $k < n - r$ or $k > n$, no EF allocation exists, and if $k = n$, one exists if and only if $K = M$. Otherwise, an EF allocation exists if and only if $R \coloneqq M \setminus K$, which is nonempty as $k < n \leq m$ and has at most $m - (n - r) = 2r$ items, is a union of $n - k$ disjoint sets of value t. Theorem 2, applied to v on the items $R \cup \{g\}$ with the pivot $\{g\}$, tells which $T \subseteq R$ have $v(T) = t$ with $O\left((r+1)^2 \log(r+2)\right)$ queries, and the rest is offline. □

Proof of Theorem 10. Let $N \coloneqq \{1, 2, \ldots, n\}$, $M \coloneqq \{1, 2, \ldots, m\}$, $s \coloneqq \min\{n, m\} \geq 2$, and $N^\star \coloneqq M^\star \coloneqq \{1, 2, \ldots, s\}$. Let $B \in \Pi_n(M)$ be given by $B_i \coloneqq M \cap \{i + qn \mid q \in \{0, 1, \ldots\}\}$ for each $i \in N$, so that $B_i \cap M^\star = \{i\}$ if $i \in N^\star$ and $B_i \cap M^\star = \emptyset$ otherwise. Let $L \coloneqq n \min\{n-1, m\}$, which is at least $n \min\{n, m\}/2$.

We first allow value queries. Let $\mathcal{J}$ be the additive instance with agents N, items M, and valuations $u_i(S) \coloneqq \mathbf{1}[i \in N^\star]\,|S \cap M^\star|$ for $S \subseteq M$ and $i \in N$. Every $A \in \mathrm{EF}(\mathcal{J})$ gives each $i \in N^\star$ at least as many items of $M^\star$ as any other agent, and some agent receives one, so $A_i \cap M^\star \neq \emptyset$ for every $i \in N^\star$; moreover, $B \in \mathrm{EF}(\mathcal{J})$. We claim that, for every $A \in \mathrm{EF}(\mathcal{J})$ and every set Q of fewer than L queries, some additive instance $\mathcal{J}'$ with agents N and items M satisfies $Q \cap \Delta(\mathcal{J}, \mathcal{J}') = \emptyset$ and $A \notin \mathrm{EF}(\mathcal{J}')$. Then no set of fewer than L queries certifies $A \in \mathrm{EF}(\mathcal{J})$ for any $A \in \mathrm{EF}(\mathcal{J})$, and Lemma 16 with $\mathcal{P}' \coloneqq \mathcal{P} \coloneqq \mathrm{EF}$ gives $\mathrm{SR}_{\mathrm{rand}}(\mathrm{EF}) \geq L$. For verification, let $\mathcal{C}'$ consist of the input $(\mathcal{J}, B)$ and, for each of the finitely many sets Q of fewer than L queries, the input $(\mathcal{J}', B)$ with the instance $\mathcal{J}'$ that the claim gives for Q and $A \coloneqq B$. A deterministic verifier that is correct on $\mathcal{C}'$ accepts $(\mathcal{J}, B)$, and if it made a set Q of fewer than L queries there, it would receive the same answers on the input $(\mathcal{J}', B)$ for Q and wrongly accept it. So it makes at least L queries on $(\mathcal{J}, B)$, and Lemma 11 gives $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EF}) \geq L$.

To prove the claim, we assign the vector $\mathbf{1}_{M|X} - \mathbf{1}_{M|Y}$ to a comparison of X with Y and the vector $\mathbf{1}_{M|X}$ to a value query of X; adding to the valuation of the queried agent any $d \in \mathbb{R}^M$ orthogonal to this vector leaves the answer unchanged. Some $i^* \in N$ receives fewer than $\min\{n-1, m\}$ of

the queries in Q, and the vectors of these queries span a subspace $H \subseteq \mathbb{R}^M$ of dimension less than $\min\{n-1, m\}$. The indicator vectors of the nonempty bundles of A, of which there are $k \geq s$, are linearly independent, so the vectors $\mathbf{1}_{M|A_j} - \mathbf{1}_{M|A_{i^*}}$ with $j \in N$ span a space of dimension $\min\{n-1, k\} \geq \min\{n-1, m\}$. Hence some $j^* \in N$ satisfies $\mathbf{1}_{M|A_{j^*}} - \mathbf{1}_{M|A_{i^*}} \notin H$, and some $d \in \mathbb{R}^M$ orthogonal to H satisfies $\langle d, \mathbf{1}_{M|A_{j^*}} - \mathbf{1}_{M|A_{i^*}} \rangle > u_{i^*}(A_{i^*}) - u_{i^*}(A_{j^*})$. Let $\mathcal{J}'$ be obtained from $\mathcal{J}$ by replacing u_{i^*} with the additive function $u_{i^*} + d$. Then $Q \cap \Delta(\mathcal{J}, \mathcal{J}') = \emptyset$, but agent i^* envies agent j^* in A at $\mathcal{J}'$, so $A \notin \mathrm{EF}(\mathcal{J}')$.

Now let only comparison queries be allowed, and let $\varepsilon \in [0, 1/2)$. For each $i \in N$, fix an injective map $r_i : M^\star \to \{0, 1, \ldots, s-1\}$ with $r_i(i) = 0$ if $i \in N^\star$, let $\delta := 2^{-1-s}$, and let $\mathcal{I}_0$ be the additive instance with agents N, items M, and valuations

$$v_i := -\delta \sum_{g \in M^\star \setminus \{i\}} 2^{r_i(g)} \mathbf{1}_{M|\{g\}} \qquad \forall i \in N. \tag{24}$$

Every agent values every bundle at most 0, and agent i values $S \subseteq M$ at 0 if and only if $S \cap M^\star \subseteq \{i\}$. Let $A \in \mathrm{EF}(\mathcal{I}_0)$. If some bundle of A contains no item of $M^\star$, every agent values it at 0, so $A_i \cap M^\star \subseteq \{i\}$ for every $i \in N$, and $A_i \cap M^\star = \{i\}$ for every $i \in N^\star$ since A is complete. Otherwise, $n \leq s$, so each of the $n = s$ bundles contains exactly one item of $M^\star$, and agent i holds item i, since otherwise she would value A_i below 0 and the bundle that contains i at 0. So every $A \in \mathrm{EF}(\mathcal{I}_0)$ satisfies $A_i \cap M^\star = \{i\}$ for $i \in N^\star$ and $A_i \cap M^\star = \emptyset$ for $i \in N \setminus N^\star$; moreover, $B \in \mathrm{EF}(\mathcal{I}_0)$, so EMPTY is not valid for $\mathcal{I}_0$.

Let P be the set of pairs $(i, g) \in N \times M^\star$, without those with $g = i$ if $n \leq m$, so that $|P| = L$. For $(i, g) \in P$ with $g \neq i$, let $\mathcal{I}_{i,g}$ be obtained from $\mathcal{I}_0$ by flipping the sign of the term of g in v_i; then agent i values g at $\delta 2^{r_i(g)} > 0$ and still values A_i at 0, so she envies agent g, who holds g, in every $A \in \mathrm{EF}(\mathcal{I}_0)$. For $(i, i) \in P$, so that $n > m$ and $i \in N^\star$, let $\mathcal{I}_{i,i}$ be obtained from $\mathcal{I}_0$ by replacing v_i with $v_i - \delta \mathbf{1}_{M|\{i\}}$; then, in every $A \in \mathrm{EF}(\mathcal{I}_0)$, agent i values A_i at $-\delta$ and envies agent $n \notin N^\star$, whose bundle she values at 0. Hence no output valid for $\mathcal{I}_0$ is valid for any $\mathcal{I}_{i,g}$, and in particular $B \notin \mathrm{EF}(\mathcal{I}_{i,g})$ for every $(i, g) \in P$. Since $\delta < 2^{-s}$, Lemma 14 applies to v_i with $c := \mathbf{0}$, $U := M^\star \setminus \{i\}$, $\sigma_g := -1$ for $g \in U$, and $r := r_i$, so $\mathcal{I}_{i,g}$ with $g \neq i$ answers a comparison differently from $\mathcal{I}_0$ only if it is a comparison to agent i whose normal h satisfies $t(h) = g$, and thus $h_g \neq 0$. Applying the same lemma to $v_i - \delta \mathbf{1}_{M|\{i\}}$, with $U := M^\star$ and $\sigma_i := -1$, and replacing σ_i by 0, which gives v_i, shows that $\mathcal{I}_{i,i}$ answers a comparison differently from $\mathcal{I}_0$ only if it is a comparison to agent i whose normal h satisfies $t(h) = i$, so that h vanishes on $M^\star \setminus \{i\}$, as $r_i(i) = 0$. So the sets $\Delta(\mathcal{I}_0, \mathcal{I}_{i,g})$ with $(i, g) \in P$ are pairwise disjoint, and they stay the same when B is attached to the instances for verification. By Lemma 12 with unit weights, every algorithm that performs the search, or the verification, correctly with probability at least $1 - \varepsilon$ makes at least $(1 - 2\varepsilon) L$ queries in expectation on $\mathcal{I}_0$, or on $(\mathcal{I}_0, B)$, respectively. □

Proof of Theorem 11. Let N, M, s, $N^\star$, $M^\star$, B, $\mathcal{J}$, u_i, r_i, and δ be as in the proof of Theorem 10, and note that $\mathcal{J}$ is monotone additive. Every $A \in \mathrm{EF}(\mathcal{J})$ gives each $i \in N^\star$ an item of $M^\star$, as shown there, so, as $|N^\star| = |M^\star|$, it gives each $i \in N^\star$ exactly one item a_i of $M^\star$ and the other agents none. We claim that, for every $A \in \mathrm{EF}(\mathcal{J})$ and every set Q of fewer than $L := s(s-1)$ queries, some monotone additive instance $\mathcal{J}'$ with agents N and items M satisfies $Q \cap \Delta(\mathcal{J}, \mathcal{J}') = \emptyset$ and $A \notin \mathrm{EF}(\mathcal{J}')$. As in the proof of Theorem 10, this gives $\mathrm{SR}_{\mathrm{rand}}(\mathrm{EF}) \geq L$ and $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EF}) \geq L$ even if value queries are allowed, and $L \geq s^2/2 = \min\{n^2, m^2\}/2$.

To prove the claim, note that some $i^* \in N^\star$ receives fewer than $s-1$ of the queries in Q, and let $H \subseteq \mathbb{R}^{M^\star}$ be the span of the restrictions to $M^\star$ of the vectors of these queries, as defined in the proof of Theorem 10. The $s-1$ vectors $\mathbf{1}_{M^\star|\{a_j\}} - \mathbf{1}_{M^\star|\{a_{i^*}\}}$ with $j \in N^\star \setminus \{i^*\}$ are linearly

independent, so one of them, say for $j = j^*$, lies outside H. Hence some $d \in \mathbb{R}^{M^\star}$ orthogonal to H satisfies $d_{a_{j^*}} > d_{a_{i^*}}$ and, after scaling, $|d_g| < 1$ for every $g \in M^\star$. Let $\mathcal{J}'$ be obtained from $\mathcal{J}$ by replacing u_{i^*} with the additive function that values each $g \in M^\star$ at $1 + d_g > 0$ and every other item at 0, which is monotone. The two valuations of i^* differ by d extended by zeros, which is orthogonal to the vectors of all queries in Q to i^*, so $Q \cap \Delta(\mathcal{J}, \mathcal{J}') = \emptyset$. But at $\mathcal{J}'$, agent i^* values A_{i^*} at $1 + d_{a_{i^*}}$ and A_{j^*} at $1 + d_{a_{j^*}}$, so $A \notin \mathrm{EF}(\mathcal{J}')$.

Now let only comparison queries be allowed, and let $\varepsilon \in [0, 1/2)$. Let $\mathcal{I}_0$ be the instance with agents N, items M, and the additive valuations $v_i \coloneqq \mathbf{1}_{M|M^\star} - \delta \sum_{g \in M^\star \setminus \{i\}} 2^{r_i(g)} \mathbf{1}_{M|\{g\}}$ for $i \in N^\star$ and $v_i \coloneqq \mathbf{0}$ for $i \in N \setminus N^\star$, which are monotone. Since $\delta \sum_{g \in M^\star} 2^{r_i(g)} < \delta 2^s = 1/2$, each $i \in N^\star$ values every $S \subseteq M$ above $|S \cap M^\star| - 1/2$ and at most $|S \cap M^\star|$, with equality if and only if $S \cap M^\star \subseteq \{i\}$. So every $A \in \mathrm{EF}(\mathcal{I}_0)$ gives each $i \in N^\star$ at least as many items of $M^\star$ as any other agent, hence, as for $\mathcal{J}$, exactly one, and the other agents none. Moreover, agent $i \in N^\star$ then holds item i, since otherwise she would value A_i below 1 and the bundle that contains i at 1. Conversely, $B \in \mathrm{EF}(\mathcal{I}_0)$, so EMPTY is not valid for $\mathcal{I}_0$.

For $i \in N^\star$ and $g \in M^\star \setminus \{i\}$, let $\mathcal{I}_{i,g}$ be obtained from $\mathcal{I}_0$ by flipping the sign of the term of g in v_i; then agent i values g at $1 + \delta 2^{r_i(g)}$ and A_i at 1, so she envies agent g, who holds g, in every $A \in \mathrm{EF}(\mathcal{I}_0)$. For $j \in N \setminus N^\star$, let $\mathcal{I}_j$ be obtained from $\mathcal{I}_0$ by replacing v_j with $\mathbf{1}_{M|\{1\}}$; then agent j envies agent 1 in every $A \in \mathrm{EF}(\mathcal{I}_0)$, as A_j contains no item of $M^\star$. These $s(s-1) + n - s$ instances are monotone additive, no output valid for $\mathcal{I}_0$ is valid for any of them, and B is EF at none of them. By Lemma 14 with $c \coloneqq \mathbf{1}_{M|M^\star}$ and the other data as in the proof of Theorem 10, the set $\Delta(\mathcal{I}_0, \mathcal{I}_{i,g})$ consists of comparisons to agent i whose normal h satisfies $\langle c, h \rangle = 0$ and $t(h) = g$, while $\Delta(\mathcal{I}_0, \mathcal{I}_j)$ consists of comparisons to agent j. So these sets are pairwise disjoint, and Lemma 12 gives both bounds as in the proof of Theorem 10.

For the bounds $(1 - 2\varepsilon)(n-1)$, fix $g^\star \in M$ and $\varepsilon \in [0, 1/2)$, and allow value queries again. Let $\mathcal{I}'_0$ be the instance with agents N and items M in which agent 1 has the valuation $\mathbf{1}_{M|\{g^\star\}}$ and every other agent has the valuation $\mathbf{0}$. For each $j \in N \setminus \{1\}$, let $\mathcal{I}'_j$ be obtained from $\mathcal{I}'_0$ by giving agent j the valuation $\mathbf{1}_{M|\{g^\star\}}$ as well; all these instances are monotone additive. Every complete allocation that gives $g^\star$ to agent 1, such as $A^\circ \coloneqq (M, \emptyset, \ldots, \emptyset)$, is EF at $\mathcal{I}'_0$. At $\mathcal{I}'_j$, every complete allocation gives $g^\star$ to some agent, and whichever of agents 1 and j does not hold it envies its holder, so $\mathrm{EF}\left(\mathcal{I}'_j\right) = \emptyset$. Hence the inputs $\mathcal{I}'_0$ and $\mathcal{I}'_j$ of decision have different correct answers, and so do the inputs $(\mathcal{I}'_0, A^\circ)$ and $\left(\mathcal{I}'_j, A^\circ\right)$ of verification. Since $\mathcal{I}'_0$ and $\mathcal{I}'_j$ differ only in the valuation of agent j, the set $\Delta\left(\mathcal{I}'_0, \mathcal{I}'_j\right)$ consists of queries to agent j, and these sets are pairwise disjoint. By Lemma 12 with unit weights, every algorithm that decides, or verifies, correctly with probability at least $1 - \varepsilon$ makes at least $(1 - 2\varepsilon)(n-1)$ queries in expectation on $\mathcal{I}'_0$, or on $(\mathcal{I}'_0, A^\circ)$, respectively. With $\varepsilon = 0$, and as a search algorithm also decides, this gives the term n of the zero-error bounds. □

Two agents and identical valuations. In this part, only comparison queries are allowed. For a finite set J of items, we call $w \in \mathbb{R}^J$ *balanced* if $\langle w, x \rangle = 0$ for some $x \in \{-1, 1\}^J$, that is, if J splits into two sets, its *halves*, of equal value. For two agents and $x \in \{-1, 1\}^M$, let $A^x \in \Pi_2(M)$ give agent 1 the items g with $x_g = 1$ and agent 2 the others, so that every complete allocation is A^x for exactly one x. Let $b \in \mathbb{R}^M$ give the items the distinct values $1, 2, 4, \ldots, 2^{m-1}$, so that $\langle b, h \rangle \neq 0$ for every $h \in \mathcal{H}_M$, as the largest power of two in the support of h exceeds the sum of all smaller ones. For $w \in \mathbb{R}^M$ and $\alpha \geq 0$, let $\mathcal{I}_\alpha(w)$ be the additive instance with two agents and the valuations $v_1 \coloneqq w + \alpha b$ and $v_2 \coloneqq w - \alpha b$, which is homogeneous if $\alpha = 0$. If $\langle w, x \rangle = 0$ and $\langle b, x \rangle > 0$, then $A^x \in \mathrm{EF}(\mathcal{I}_\alpha(w))$ for every $\alpha \geq 0$, as $v_1(A^x_1) - v_1(A^x_2) = \langle w + \alpha b, x \rangle = \alpha \langle b, x \rangle$ and $v_2(A^x_2) - v_2(A^x_1) = -\langle w - \alpha b, x \rangle = \alpha \langle b, x \rangle$. For w with positive entries, let $\alpha(w) > 0$ be the least

ratio $|\langle w,h\rangle| / |\langle b,h\rangle|$ over all $h \in \mathcal{H}_M$ with $\langle w,h\rangle \neq 0$. If $0 < \alpha < \alpha(w)$, then, for every $h \in \mathcal{H}_M$, the sign of $\langle w \pm \alpha b, h\rangle$ is that of $\langle w,h\rangle$ if $\langle w,h\rangle \neq 0$, by the choice of $\alpha(w)$, and that of $\pm\langle b,h\rangle$ otherwise, so $\mathcal{I}_\alpha(w)$ is non-degenerate and its item values are positive. If $\alpha = 0$ or $0 < \alpha < \alpha(w)$, then A^x is EF in $\mathcal{I}_\alpha(w)$ only if $\langle w,x\rangle = 0$, since EF requires $\langle w+\alpha b, x\rangle \geq 0 \geq \langle w-\alpha b, x\rangle$, whereas both have the sign of $\langle w,x\rangle$ if it is nonzero; hence $\mathcal{I}_\alpha(w)$ then has an EF allocation if and only if w is balanced.

The following lemma gives the bounds of order m^2 in Theorems 12 and 13, by a reduction from the knapsack problem.

Lemma 17. *Let $k \geq 2$, $m \coloneqq n+k-2$, and $\varepsilon \in [0,1/2)$, and let $\mathcal{C}$ be the set of homogeneous additive instances with n agents and m items whose item values are all positive or, if $n=2$, the set of monotone non-degenerate additive instances with two agents and m items. Then $\mathrm{DC}_\varepsilon(\mathrm{EF}) \geq ck^2$ for some $c > 0$ that depends only on ε, and, if $k \geq 3$, the same holds for $\mathrm{SR}_\varepsilon(\mathrm{EF})$ with $\mathcal{C}^{\mathrm{EF}}$ in place of $\mathcal{C}$.*

Proof. Let $K \subseteq M$ be a set of k items, the *core*, and $E \coloneqq M \setminus K$, so that $|E| = n-2$, and $E = \emptyset$ for the non-degenerate set. For $u \in \mathbb{R}^K$ with positive entries, let $\hat{u} \in \mathbb{R}^M$ agree with u on K and give each item of E the value $u(K)/2$. An EF allocation of the homogeneous instance with the valuation $\hat{u}$ gives every bundle the average value $\hat{u}(M)/n = u(K)/2$, so, as item values are positive, every item of E forms a bundle by itself, and the other two bundles are the halves of a split of K; conversely, every such allocation is EF, so an EF allocation exists if and only if u is balanced.

Let $d \coloneqq \lfloor (k-2)/2 \rfloor$, suppose that $d \geq 1$, and let K consist of items p_j and q_j for $j \in \{1,\dots,d\}$, items r and r', and, if $k = 2d+3$, an item r''. For $z \in \mathbb{R}^d$, let $\kappa \coloneqq 3 + 2\sum_j |z_j|$, $B_j \coloneqq \kappa 4^{j-1}$, $\theta \coloneqq 1 + \sum_j B_j$, $W \coloneqq \sum_j (2B_j + z_j)$, and $\Gamma \coloneqq 2W + 2\theta + 1$, and let $u(z) \in \mathbb{R}^K$ give p_j the value $B_j + z_j$, q_j the value B_j, and r' the value $\Gamma - W + \theta$, and give r the value $\Gamma - \theta$ if $k = 2d+2$, and $3\Gamma - \theta$ if $k = 2d+3$, in which case r'' has the value 2Γ. These values are positive, as $B_j \geq \kappa > |z_j|$ and $\Gamma > W + \theta$, and their sum is 2Γ if $k = 2d+2$ and 6Γ otherwise. The item r together with r' has the value $2\Gamma - W > \Gamma$, or $4\Gamma - W > 3\Gamma$, and together with r'' the value $5\Gamma - \theta > 3\Gamma$, so a half of K that contains r consists of r and a set P of items p_j and q_j of value θ, and conversely, for every such P, the set $P \cup \{r\}$ is a half. If P contains $c_j \in \{0,1,2\}$ items of each pair $\{p_j, q_j\}$, and p_j exactly for $j \in S$, then its value is θ if and only if $\kappa \sum_j (c_j - 1) 4^{j-1} = 1 - \sum_{j \in S} z_j$. The right-hand side has absolute value less than κ, so the integer $\sum_j (c_j - 1) 4^{j-1}$ vanishes, which forces $c_j = 1$ for every j, as its largest nonzero term would exceed the sum of the others in absolute value, and the condition then reads $\sum_{j \in S} z_j = 1$; conversely, if $\sum_{j \in S} z_j = 1$, the set $\{p_j \mid j \in S\} \cup \{q_j \mid j \notin S\}$ has the value θ. So $u(z)$ is balanced if and only if z lies in $\mathrm{KS}_d \coloneqq \{z \in \mathbb{R}^d \mid \langle a,z\rangle = 1 \text{ for some } a \in \{0,1\}^d\}$. For the homogeneous set, let $\mathcal{I}(z) \in \mathcal{C}$ be the instance with the valuation $\widehat{u(z)}$, and, for the non-degenerate one, where $M = K$, let $\mathcal{I}(z) \coloneqq \mathcal{I}_\alpha(u(z)) \in \mathcal{C}$ for any α with $0 < \alpha < \alpha(u(z))$, as all these instances answer every comparison alike by the sign rule above. In both cases, $\mathcal{I}(z)$ has an EF allocation if and only if $z \in \mathrm{KS}_d$. On the set of z with given signs σ_j of the entries z_j, the entries of $\widehat{u(z)}$ are affine functions of z, as $|z_j| = \sigma_j z_j$, so the answer to a comparison of bundles X and Y for a given agent at $\mathcal{I}(z)$ is determined by the sign of the affine function $\left\langle \widehat{u(z)}, \mathbf{1}_{M|X} - \mathbf{1}_{M|Y} \right\rangle$ of z, where $\widehat{u(z)} \coloneqq u(z)$ for the non-degenerate set, whose valuations $u(z) \pm \alpha b$ break ties by the sign rule above, independently of z.

Now let Λ decide EF on $\mathcal{C}$ or, for promise search, return an EF allocation on every instance of $\mathcal{C}^{\mathrm{EF}}$, in each case correctly with probability at least $1-\varepsilon$ and with at most Q queries in expectation on every instance of $\mathcal{C}$, or of $\mathcal{C}^{\mathrm{EF}}$, respectively, and let $L \coloneqq \lceil 4Q/(1-2\varepsilon) \rceil$. For each value s of

the internal randomness, let D_s be the decision tree that runs Λ_s for at most L queries, where a run that makes no further query without halting becomes a leaf, and answers YES if and only if Λ_s answers YES within them or, for search, returns within them an allocation $A \in \Pi_n(M)$ in which every item of E forms a bundle by itself and whose two other bundles A_i and A_j satisfy $v_i(A_j) \le v_i(A_i)$ and $v_j(A_i) \le v_j(A_j)$, which D_s checks with two more comparisons. Every EF allocation of $\mathcal{I}(z)$ passes this check, and an allocation that passes it is EF if $n = 2$ and splits K into halves if $n > 2$, where the valuations are identical, so one passes it only if $\mathcal{I}(z)$ has an EF allocation. Hence, for search, D_s answers correctly on every $\mathcal{I}(z)$ without an EF allocation, whatever Λ_s does there, and on the other $\mathcal{I}(z)$, as on every $\mathcal{I}(z)$ for decision, D_s answers correctly unless Λ_s fails or makes more than L queries, which, by Markov's inequality, happens with probability at most $\varepsilon + Q/(L+1) \le (1+2\varepsilon)/4$. As there are finitely many comparisons, D_s is one of finitely many binary decision trees of depth at most $L+2$. Testing the signs of $z_1, \dots, z_d$ first and then replacing each comparison of D_s by a test of the sign of the corresponding affine function of z gives a ternary decision tree D'_s of depth at most $d+L+2$ that accepts z if and only if D_s answers YES at $\mathcal{I}(z)$. Rounding the probabilities of the finitely many trees D'_s to multiples of $1/N$ for a large N, we obtain a uniformly random member of a finite family of such trees that decides whether $z \in \mathrm{KS}_d$ with error probability at most $\gamma := (3+2\varepsilon)/8 < 1/2$ for every $z \in \mathbb{R}^d$, which is the form the bound below requires: a finite family of trees and a fixed error probability below $1/2$. By Grigoriev et al. [81, Section 2 and Corollary 2(2)], every such family for KS_d, whose nodes test the signs of affine functions of z, contains a tree of depth at least $c_\gamma d^2$ if $d \ge d_\gamma$, where $c_\gamma > 0$ and d_γ depend only on γ. As $L < 4Q/(1-2\varepsilon) + 1$ and $d \ge (k-3)/2$, this gives $Q > (1-2\varepsilon)\left(c_\gamma d^2 - d - 3\right)/4 \ge ck^2$ for all $k \ge k_\varepsilon$, for some $c > 0$ and $k_\varepsilon \ge 4$ that depend only on ε.

For $2 \le k < k_\varepsilon$, after decreasing c, it suffices that $Q \ge 1 - 2\varepsilon$, which Lemma 12 gives for any two instances of $\mathcal{C}$, or of $\mathcal{C}^{\mathrm{EF}}$, that have disjoint sets of valid outputs. For $g \in K$ and $a \in \{k-1, k\}$, let $u_{g,a} \in \mathbb{R}^K$ give g the value a and the other core items the value 1; then $u_{g,k-1}$ is balanced, with the halves $\{g\}$ and $K \setminus \{g\}$ only, and $u_{g,k}$ is not, as its entries are integers with an odd sum. So, with the valuations $\hat{u}_{g,a}$, or with $\mathcal{I}_\alpha(u_{g,a})$ for small $\alpha > 0$, the instances for $u_{g,k-1}$ and $u_{g,k}$ serve for decision, and, if $k \ge 3$, those for $u_{g,k-1}$ and $u_{g',k-1}$ with distinct $g, g' \in K$ serve for search, as every EF allocation of the former has the bundle $\{g\}$, whereas every EF allocation of the latter puts g in a bundle with the $k-2 \ge 1$ other items of $K \setminus \{g'\}$. □

Proof of Theorem 12. A search algorithm also decides, by answering that an EF allocation exists if and only if it returns an allocation, with the same cost and at most the same error probability, so $\mathrm{DC}_\varepsilon(\mathrm{EF}) \le \mathrm{SR}_\varepsilon(\mathrm{EF})$ for every $\varepsilon \in [0, 1/2)$, and Lemma 17 with $n = 2$ and $k = m$ gives the bounds. □

Proof of Theorem 13. Let $k := m - n + 2 \ge 2$. As a search algorithm also decides, and $k \ge 3$ if and only if $m > n$, Lemma 17 gives the bounds $\Omega\left(k^2\right)$. For the sharp quadratic term, we use the argument of Dobkin and Lipton [59, p. 414]: an algorithm must tell apart many regions of instances without an EF allocation, as its leaves are convex. Let K, E, and $\hat{u} \in \mathbb{R}^M$ for $u \in \mathbb{R}^K$ with positive entries be as in the proof of Lemma 17, so that $|K| = k$, and let $\mathcal{I}(u) \in \mathcal{C}$ be the instance with the common valuation $\hat{u}$, which has an EF allocation if and only if u is balanced. The *pattern* of a positive $w \in \mathbb{R}^K$ that is not balanced is $(\operatorname{sign}\langle w, x\rangle)_{x \in \{-1,1\}^K}$, and we let P_k be the number of patterns; the positive vectors with a given pattern form a nonempty open set. Let $R(d)$ be the number of threshold functions on $\{-1,1\}^d$, that is, of the maps $y \mapsto \operatorname{sign}(\langle a, y\rangle + a_0)$ with $a \in \mathbb{R}^d$ and $a_0 \in \mathbb{R}$ that vanish nowhere on $\{-1,1\}^d$. Fix $g^* \in K$, and extend each $y \in \{-1,1\}^{K \setminus \{g^*\}}$ to $\hat{y} \in \{-1,1\}^K$ with $\hat{y}_{g^*} := 1$. Every threshold function on $\{-1,1\}^{K \setminus \{g^*\}}$ is $y \mapsto \operatorname{sign}\langle w, \hat{y}\rangle$ for some $w \in \mathbb{R}^K$ without zero entries, as its defining strict inequalities are open, and then $\langle w, x\rangle \ne 0$ for every

$x \in \{-1, 1\}^K$, as $x = \pm\hat{y}$ for some y. With $\tau_g \coloneqq \operatorname{sign} w_g$ and $\tau x \coloneqq (\tau_g x_g)_{g \in K}$, the vector $w' \coloneqq \tau w$ is positive and not balanced, as $\langle w', x\rangle = \langle w, \tau x\rangle$, so the function, which is $y \mapsto \operatorname{sign}\langle w', \tau\hat{y}\rangle$, is determined by τ and the pattern of w'. Hence $R(k-1) \leq 2^k P_k$, and $\log_2 R(d) \geq d^2 - d\log_2 d - O(d)$ by Kahn et al. [99, Section 4], who derive this from their Corollary 4(b) by an argument that they attribute to Zuev [162] (see also Zuev [161]), so $\log_2 P_k \geq k^2 - k\log_2 k - O(k) = k^2 - O(k\log k)$. For each pattern, fix a positive rational vector with this pattern, and, for each two of these vectors w and w' and each x at which their patterns differ, let u be the point of the segment between w and w' with $\langle u, x\rangle = 0$, which is positive, rational, and balanced. Then the instances $\mathcal{I}(u)$ have EF allocations, and the P_k instances $\mathcal{I}(w)$ do not. Let a deterministic algorithm decide EF correctly on the countable set of instances in $\mathcal{C}$ with rational values. The instances that answer a comparison TRUE, as well as those that answer it FALSE, form a convex set of valuation tables, and $w \mapsto \hat{w}$ is linear, so the set of w for which $\mathcal{I}(w)$ follows a given path of the algorithm is convex. Hence, if $\mathcal{I}(w)$ and $\mathcal{I}(w')$ ended at the same leaf, then so would $\mathcal{I}(u)$ for some u as above, with the wrong answer that no EF allocation exists. So the P_k instances $\mathcal{I}(w)$ end at distinct leaves, and Lemma 11 gives $\mathrm{DC}_{\mathrm{rand}}(\mathrm{EF}) \geq \log_2 P_k$, and so does $\mathrm{SR}_{\mathrm{rand}}(\mathrm{EF})$.

It remains to show $\mathrm{DC}_{\mathrm{rand}}(\mathrm{EF}) \geq n$, and we identify each instance of $\mathcal{C}$ with its common valuation $w \in \mathbb{R}^M$. For $A \in \Pi_n(M)$, let E_A be the set of $w' \in \mathbb{R}^M$ with $w'(A_1) = \cdots = w'(A_n)$. A positive w has an EF allocation if and only if it lies in E_A for some A with nonempty bundles, as the bundles of an EF allocation have the same positive value, and E_A is a subspace of dimension $m-n+1$, as the vectors $\mathbf{1}_{M|A_i} - \mathbf{1}_{M|A_n}$ with $i < n$ are linearly independent. Fix a rational positive w_0 with an EF allocation, for example with $n-1$ items of value $m-n+1$ and the other items of value 1, and let a deterministic algorithm decide EF correctly on the countable set of instances in $\mathcal{C}$ with rational values. Suppose that its run on w_0 makes at most $n-1$ comparisons of distinct bundles, say $[\langle w, h_\ell\rangle \leq 0]$ with $h_\ell \in \mathcal{H}_M$, and let T be the set of the ℓ with $\langle w_0, h_\ell\rangle = 0$; the other comparisons have the same answers in a neighborhood of w_0. If the h_ℓ with $\ell \in T$ span a space of dimension at most $n-2$, let U be the set of the points $w_0 + \delta$ near w_0 with $\langle \delta, h_\ell\rangle = 0$ for $\ell \in T$, which lie in an affine subspace of dimension at least $m-n+2$ spanned by rational vectors. Otherwise, all $n-1$ comparisons lie in T with linearly independent normals, so some rational δ_0 satisfies $\langle \delta_0, h_\ell\rangle = -1$ for every ℓ, and we let U be a small open ball around $w_0 + \lambda\delta_0$ for a small rational $\lambda > 0$. In both cases, every rational point of U is a positive instance on which the run is that on w_0, so the algorithm answers YES there, and these points would lie in the finitely many subspaces E_A of dimension $m-n+1$; this is impossible, as such subspaces cannot cover the rational points of a relatively open subset of a rational affine space of larger dimension. So the algorithm makes at least n queries on w_0, and Lemma 11 gives $\mathrm{DC}_{\mathrm{rand}}(\mathrm{EF}) \geq n$.

For negative item values, negating all values maps the instances above onto those with negative values; it preserves EF, as an allocation of a homogeneous instance is EF exactly when its bundles have equal values, and it turns each comparison of X with Y into one of Y with X, so all the bounds carry over. □

Proof of Remark 6. Let $m = n$, and let $v \in \mathbb{R}^M$ also denote the common valuation, with entries v_g. In an EF allocation, every bundle has the positive value $v(M)/n$, so every bundle is nonempty and hence a single item, and all items have the same value; conversely, if they do, every bijection between agents and items is EF. Let v° have all entries equal, and let a deterministic algorithm decide EF correctly on the countable set of instances $v \in \mathcal{C}$ with $v - v^\circ \in \mathbb{Q}^M$. In its run on v°, a comparison with a normal $h \in \mathcal{H}_M$ is answered TRUE with $\langle v^\circ, h\rangle = 0$ if $\sum_g h_g = 0$, and otherwise $\langle v^\circ, h\rangle \neq 0$, so its answer does not change in a neighborhood of v°. Suppose that the run makes at most $n-1$ queries, and let H_0 be the set of the normals of the first kind, which lie in the $(n-1)$-dimensional space $U \coloneqq \left\{\delta \in \mathbb{R}^M \mid \sum_g \delta_g = 0\right\}$. Either H_0 does not span U, and some

nonzero $\delta \in U$ is orthogonal to H_0, or H_0 is a basis of U, and some $\delta \in U$ has $\langle \delta, h \rangle = -1$ for every $h \in H_0$; in both cases, δ is nonzero and can be chosen rational. For a small rational $\lambda > 0$, the instance $v^\circ + \lambda\delta$ lies in $\mathcal{C}$, has items of different values, and answers all comparisons of the run as v° does, so the algorithm answers wrongly on it. Hence every such algorithm makes at least n queries on v°, and so does almost every Λ_s of a zero-error randomized algorithm Λ, since almost every Λ_s is correct on this countable set, as in the proof of Lemma 11. Conversely, the n comparisons of $v(\{g_\ell\})$ with $v(\{g_{\ell+1}\})$, for an ordering $g_1, \ldots, g_n$ of M and $g_{n+1} \coloneqq g_1$, all answer TRUE if and only if all items have the same value.

For the tester, $v(S) - v(\pi(S)) = \sum_{g \in S} \left(v_g - v_{\pi(g)}\right)$, and if the items do not all have the same value, then $v_g \neq v_{\pi(g)}$ for some g, as π is a single cycle. Given which other items lie in S, the two choices for g give two different values of this difference, so $v(S) = v(\pi(S))$ with probability at most $1/2$, and $\lceil \log_2(1/\varepsilon) \rceil$ independent rounds all fail to reject with probability at most ε. If all items have the same value, no round rejects, as $|\pi(S)| = |S|$, and every bijection is EF; so the tester, with two comparisons per round, decides EF with one-sided error at most ε, and returning a bijection upon acceptance, and EMPTY otherwise, solves search with the same one-sided error. □

Three or more agents. We prove Theorem 14 first for monotone non-degenerate instances. Let $n \geq 3$ and $k \coloneqq m - n + 1 \geq 4$, and let the items be a *core* K of k items with a distinguished item $t \in K$, an item a, and items $c_3, \ldots, c_n$, with $D \coloneqq \{c_3, \ldots, c_n\}$. Let $r \in \mathbb{R}^M$ give the items the distinct values $1, 2, 4, \ldots, 2^{m-1}$, and let $\tilde{r} \in \mathbb{R}^M$ agree with r on D and with $-r$ on $K \cup \{a\}$; then $\langle r, h \rangle \neq 0$ and $\langle \tilde{r}, h \rangle \neq 0$ for every $h \in \mathcal{H}_M$, since the largest power of two in the support of h exceeds the sum of all smaller ones. For $w \in \mathbb{R}^K$ with positive entries, let $\bar{w} \in \mathbb{R}^M$ agree with w on K and vanish elsewhere, $W \coloneqq \sum_{g \in K} w_g$, $\mu \coloneqq \min\left\{|\langle w, h \rangle| \mid h \in \{-1, 0, 1\}^K, \langle w, h \rangle \neq 0\right\} \leq W$, $\Gamma \coloneqq 2W$, and $\delta \coloneqq \mu / 2^{m+1}$. Let $\mathcal{I}(w)$ be the additive instance with the valuations

$$v_1 \coloneqq \Gamma \mathbf{1}_{M|\{t,a\}} + \bar{w} + \delta r, \qquad v_2 \coloneqq \Gamma \mathbf{1}_{M|\{t,a\}} + \bar{w} + \delta \tilde{r}, \tag{25}$$

and, for $j \geq 3$, the valuation v_j obtained from r by exchanging the values of c_j and of the item of value 2^{m-1}, so that $v_j(\{c_j\}) > v_j(M \setminus \{c_j\})$. For $h \in \mathcal{H}_M$, let $\ell(h) \coloneqq h_t + h_a$, and let $h_K \in \{-1, 0, 1\}^K$ be its restriction to K.

Lemma 18. *For $h \in \mathcal{H}_M$ and $i \in \{1, 2\}$, the sign of $\langle v_i, h \rangle$ in $\mathcal{I}(w)$ is that of $\ell(h)$ if $\ell(h) \neq 0$, that of $\langle w, h_K \rangle$ if $\ell(h) = 0 \neq \langle w, h_K \rangle$, and that of $\langle r, h \rangle$ or $\langle \tilde{r}, h \rangle$, respectively, otherwise; and $\langle v_j, h \rangle \neq 0$ for $j \geq 3$. Hence $\mathcal{I}(w)$ is monotone and non-degenerate, and its type at a coordinate (i, h) does not depend on w unless $i \in \{1, 2\}$, $\ell(h) = 0$, and $h_K \neq \mathbf{0}$, in which case it depends on w only through the sign of $\langle w, h_K \rangle$.*

Proof. We have $\langle v_i, h \rangle = \Gamma \ell(h) + \langle w, h_K \rangle + \delta\rho$ with $\rho \in \{\langle r, h \rangle, \langle \tilde{r}, h \rangle\}$, where $|\langle w, h_K \rangle| \leq W$ and $0 < |\delta\rho| < \delta 2^m = \mu/2 \leq W/2$. So the first term decides the sign if it is nonzero, as $\Gamma = 2W$; otherwise the second does if it is nonzero, as its absolute value is then at least μ; and otherwise the third does. Each v_j with $j \geq 3$ is a permutation of r. For $h \coloneqq \mathbf{1}_{M|\{g\}}$, the three cases give positive values to $g \in \{t, a\}$, to $g \in K \setminus \{t\}$, and to $g \in D$, so $\mathcal{I}(w)$ is monotone; it is non-degenerate since no sign is 0. □

Lemma 19. *If $A \in \mathrm{EF}(\mathcal{I}(w))$, then $h \coloneqq \left(\mathbf{1}_{M|A_1} - \mathbf{1}_{M|A_2}\right)_K$ satisfies $h_t \neq 0$ and $\langle w, h \rangle = 0$. Conversely, if some $h \in \{-1, 0, 1\}^K$ satisfies $h_t \neq 0$ and $\langle w, h \rangle = 0$, then $\mathrm{EF}(\mathcal{I}(w)) \neq \emptyset$.*

Proof. For $j \geq 3$, agent j envies the holder of c_j unless $c_j \in A_j$, in which case she envies nobody. Let A be EF and $g \coloneqq \mathbf{1}_{M|A_1} - \mathbf{1}_{M|A_2}$, so that $\langle v_1, g \rangle \geq 0 \geq \langle v_2, g \rangle$. By Lemma 18, these signs

would agree, which is impossible, unless $\ell(g) = 0$ and $\langle w, g_K \rangle = 0$, as also holds if $g = \mathbf{0}$; so $\langle w, h \rangle = 0$, and A_1 and A_2 contain the same number $\nu \in \{0, 1\}$ of the items t and a. If $\nu = 0$, the bundle A_j containing t satisfies $\ell\left(\mathbf{1}_{M|A_j} - \mathbf{1}_{M|A_1}\right) \geq 1$, so agent 1 envies it; hence $\nu = 1$, and t lies in exactly one of A_1 and A_2, that is, $h_t \neq 0$. Conversely, let $h_t = 1$ without loss of generality, and let $A_1 \coloneqq \{g \in K \mid h_g = 1\}$, $A_2 \coloneqq \{g \in K \mid h_g = -1\} \cup \{a\}$, $A_3 \coloneqq (K \setminus (A_1 \cup A_2)) \cup \{c_3\}$, and $A_j \coloneqq \{c_j\}$ for $j \geq 4$. The vector $g \coloneqq \mathbf{1}_{M|A_1} - \mathbf{1}_{M|A_2}$ has $\ell(g) = 0$, $g_K = h$, and $\langle \tilde{r}, g \rangle = -\langle r, g \rangle$, so by Lemma 18 agents 1 and 2 strictly disagree on A_1 against A_2, and after exchanging A_1 and A_2 if necessary, each strictly prefers her own bundle. They envy no other bundle by Lemma 18, as $\ell = 1$ for their bundle against any other, and the other agents hold their items c_j. □

For $L \in \{0, 1, \ldots\}$, let $J_L \coloneqq \left(2^L - (-1)^L\right)/3$, so that $J_0 = 0$, $J_1 = 1$, $J_L = J_{L-1} + 2J_{L-2}$ for $L \geq 2$, and J_L is odd for $L \geq 1$. Following Chevyrev et al. [47, Theorem 3], let $z^* \in \mathbb{Z}^K$ take the values $2, 4, \ldots, 2^{k-1}$ on $K \setminus \{t\}$ and J_{k-1} at t, let $\mathcal{H}^* \coloneqq \left\{h \in \{-1, 0, 1\}^K \mid \langle z^*, h \rangle = 1\right\}$, and let $y_h \coloneqq z^* - h/\left|\operatorname{supp} h\right|$ for $h \in \mathcal{H}^*$.

Lemma 20. *The vector z^* satisfies $\langle z^*, h \rangle \neq 0$ for every nonzero $h \in \{-1, 0, 1\}^K$, and $|\mathcal{H}^*| = F_{k+1}$. Each $h \in \mathcal{H}^*$ satisfies $h_t \neq 0$, and y_h is a positive vector in the closure of the chamber of z^* that lies on the hyperplane with normal h and on no other hyperplane of the ternary arrangement on K.*

Proof. The subsets of $K \setminus \{t\}$ have distinct even sums under z^*, and adding t makes them odd, so all 2^k subset sums are distinct. For $h \in \mathcal{H}^*$, the entry h_t is nonzero because 1 is odd and all other values are even, and $s \coloneqq |\operatorname{supp} h| \geq 2$, as a single entry gives $|\langle z^*, h \rangle| \geq 2$. Then $\langle y_h, h \rangle = 1 - s/s = 0$, and every entry of y_h is at least $2 - 1/s > 0$. For nonzero $g \in \{-1, 0, 1\}^K$ with $g \neq \pm h$, oriented so that $\langle z^*, g \rangle \geq 1$, we have $\langle y_h, g \rangle = \langle z^*, g \rangle - \langle g, h \rangle / s \geq 0$, with equality only if $\langle z^*, g \rangle = 1$ and $g = h$ on $\operatorname{supp} h$; but then $g - h \in \{-1, 0, 1\}^K$ is supported off $\operatorname{supp} h$ with $\langle z^*, g - h \rangle = 0$, so $g = h$, a contradiction. Hence y_h lies on no other hyperplane, and the segment from y_h to z^* minus y_h lies in the chamber of z^*.

For the count, let $\rho_L(x)$ be the number of $c \in \{-1, 0, 1\}^L$ with $\sum_{i<L} c_i 2^i = x$, so that $\rho_L(-x) = \rho_L(x)$, $\rho_0(x) = [x = 0]$, and, by the last digit, $\rho_L(x) = \rho_{L-1}(x/2)$ for even x and $\rho_L(x) = \rho_{L-1}\left((x-1)/2\right) + \rho_{L-1}\left((x+1)/2\right)$ for odd x. Writing the entries of $h \in \mathcal{H}^*$ at the values 2^{i+1} as c_i, the condition reads $\sum_{i<k-1} c_i 2^i = \left(1 - h_t J_{k-1}\right)/2$ with $h_t = \pm 1$, so $|\mathcal{H}^*| = S_{k-1}$, where $S_L \coloneqq \rho_L\left((J_L - 1)/2\right) + \rho_L\left((J_L + 1)/2\right)$. As $J_L = J_{L-1} + 2J_{L-2}$ and $J_{L-1} - 2J_{L-2} = (-1)^L$, the two arguments are J_{L-1} and $2J_{L-2}$ for $L \geq 2$. For $L \geq 3$, the odd J_{L-1} gives $\rho_L\left(J_{L-1}\right) = S_{L-1}$, and $\rho_L\left(2J_{L-2}\right) = \rho_{L-1}\left(J_{L-2}\right) = S_{L-2}$ likewise, so $S_L = S_{L-1} + S_{L-2}$. With $S_1 = 2$ and $S_2 = 3$, this gives $S_L = F_{L+2}$ and $|\mathcal{H}^*| = F_{k+1}$. □

Proof of Theorem 14. First let $\mathcal{C}$ be the set of monotone non-degenerate additive instances, and $F = F_{m-n+2} = F_{k+1}$. Let $\mathcal{I}^* \coloneqq \mathcal{I}\left(z^*\right)$ and $\mathcal{I}_h \coloneqq \mathcal{I}\left(y_h\right)$ for $h \in \mathcal{H}^*$, which lie in $\mathcal{C}$ by Lemma 18. By Lemmas 19 and 20, $\mathrm{EF}\left(\mathcal{I}^*\right) = \emptyset$ and $\mathrm{EF}\left(\mathcal{I}_h\right) \neq \emptyset$. For $g \in \{-1, 0, 1\}^K$ with $g \neq \pm h$, the signs of $\langle z^*, g \rangle$ and $\langle y_h, g \rangle$ agree by Lemma 20, so by Lemma 18, the types of $\mathcal{I}^*$ and $\mathcal{I}_h$ differ only at coordinates (i, g) with $i \in \{1, 2\}$, $\ell(g) = 0$, and $g_K = \pm h$, which we call the coordinates of h. The coordinates of distinct $h, h' \in \mathcal{H}^*$ are distinct, as $-h \notin \mathcal{H}^*$. So every emptiness certificate at $\mathcal{I}^*$ contains a coordinate of each $h \in \mathcal{H}^*$, which gives $E_\emptyset\left(\mathrm{EF}\right) \geq F$. Moreover, the sets $\Delta\left(\mathcal{I}^*, \mathcal{I}_h\right)$ consist of comparison queries at coordinates of h and are thus pairwise disjoint, and no output valid for $\mathcal{I}^*$, which is NO or EMPTY, is valid for any $\mathcal{I}_h$. No allocation is EF in two distinct $\mathcal{I}_h$ and $\mathcal{I}_{h'}$ either: by Lemma 19, its vector h'' of Lemma 19 would be $\pm h$ and $\pm h'$, as y_h and $y_{h'}$ lie on no other hyperplane, so $h = h'$. So Lemma 12 gives the bound $(1 - 2\varepsilon) F$ for decision and, as

the sets of valid outputs of search at $\mathcal{I}^*$ and at the $\mathcal{I}_h$ are pairwise disjoint, the bound $(1-\varepsilon)F-\varepsilon$ for search.

For $\mathcal{C}^{\mathrm{EF}}$, recall that no allocation is EF in two distinct $\mathcal{I}_h$. Fix the internal randomness of an algorithm and run it on $\mathcal{I}^*$, on which nothing is required of it, and on every $\mathcal{I}_h$. Let $T \in \{0,1,\dots,\infty\}$ be its number of queries on $\mathcal{I}^*$ and q_h that on $\mathcal{I}_h$, let d be the number of h such that the run on $\mathcal{I}^*$ makes a query at a coordinate of h, and let R be the number of h for which the run on $\mathcal{I}_h$ returns an EF allocation. For each of the $F-d$ other h, the runs on $\mathcal{I}_h$ and $\mathcal{I}^*$ coincide, so $q_h = T \geq d$ and they have the same output, which is EF in at most one of them; hence $R \leq d+1$. Ordering the other d values of h by their first query in the run on $\mathcal{I}^*$, the run on the j-th of them coincides with that on $\mathcal{I}^*$ up to that query, which is at least the j-th, so $q_h \geq j$. Summing, $\sum_h q_h \geq d(d+1)/2 + (F-d)d \geq (F+2)(R-1)/2$, as $d(2F-d+1) \geq d(F+2)$ if $d \leq F-1$ and $F(F+1) \geq (F-1)(F+2)$ if $d = F$. A deterministic algorithm that succeeds on every $\mathcal{I}_h$ has $R = F$, so $d \geq F-1$, and the $(F-1)$-st of the queried h costs at least $F-1$ queries. For a randomized algorithm that succeeds with probability at least p on every $\mathcal{I}_h$ with expected cost at most Q, taking expectations gives $FQ \geq (F+2)(pF-1)/2$.

For the set of monotone additive instances and $F = F_m$, let $k \coloneqq m-1 \geq 4$, drop the items c_j and the terms with δ, and give agents $j \geq 3$ the zero valuation. Then the sign of $\langle v_1, h\rangle = \langle v_2, h\rangle$ is that of $\ell(h)$ if $\ell(h) \neq 0$ and that of $\langle w, h_K\rangle$ otherwise, and Lemma 19 holds with the same proof, where agents 1 and 2 now value A_1 and A_2 equally and $A_3 \coloneqq K \setminus (A_1 \cup A_2)$ and $A_j \coloneqq \emptyset$ for $j \geq 4$; the rest of the proof applies verbatim. □

Proof of Remark 7. Let $k \coloneqq m-n+1 \geq 4$, and let C be a chamber of the ternary arrangement on K with $F \coloneqq f(k)$ facets; as sign changes of coordinates preserve the arrangement, we may assume that C lies in the positive orthant. By the proof of Lemma 13, each facet, with normal h, contains a point y that lies on no other hyperplane; if h is not $\pm\mathbf{1}_{K\setminus\{g\}}$ for any g, then y lies on no coordinate hyperplane and is thus positive. For such h with $h_t \neq 0$, the proof of Theorem 14 applies with any $z^* \in C$ and these points in place of the y_h, so $\mathrm{DC}_{\mathrm{rand}}(\mathrm{EF})$ is at least the number N_t of these facets. A facet whose normal has a support of size s separates 2^{k-s} pairs of bundles that are consecutive in the order of 2^K at C, and distinct facets separate distinct pairs [47, Proposition 1 and the proof of Lemma 4], so $\sum_h 2^{-s_h} < 1$, and Jensen's inequality gives $\sum_h s_h \geq F\log_2 F$. At most one facet has a normal of support 1 [47, Proposition 1], so $\sum_{t\in K} N_t \geq F\log_2 F - 1$, and some t has $N_t \geq (F\log_2 F - 1)/k \geq F/2$, since $F \geq F_{k+1}$ by Eq. (2) gives $\log_2 F - 1/F \geq \log_2 F_{k+1} - 1/F_{k+1} \geq k/2$ for $k \geq 4$. The other claims follow from Corollary 4 and Theorem 1, the latter with $E(\mathrm{EF}) \leq n(n-1)$ and $\log E_\emptyset(\mathrm{EF}) = \Omega(m)$. □

B.2 Section 3.2

Proof of Theorem 15. First, suppose that $\mathcal{C}$ consists of instances with two agents, and let $\mathcal{I} \in \mathcal{C}$ be the unknown instance, where $\mathcal{I} = (\{1,2\}, M, (v_1, v_2))$.

We compute an allocation $A \in \mathrm{EF1}(\mathcal{I})$ using at most $\lceil \log_2(|M|-1)\rceil + 3$ queries by cut and choose (cf. [127, Theorem 3.1] and [28, Theorem 1]). Write $M = \{g_1, g_2, \dots, g_{|M|}\}$, let $X_k \coloneqq \{g_1, g_2, \dots, g_k\}$ and $Y_k \coloneqq M \setminus X_k$ for each $k \in \{0,1,\dots,|M|\}$, and let q_k be the answer of the comparison query whether $v_1(Y_{k+1}) \leq v_1(X_k)$ for each $k \in \{0,1,\dots,|M|-1\}$. If $q_{k-1} \neq q_k$ for some $k \in \{1,2,\dots,|M|-1\}$, let $(X,Y) \coloneqq (X_k, Y_k)$; then $R \coloneqq \{g_k\}$ or $R \coloneqq \{g_{k+1}\}$ satisfies the condition in Definition 6 for agent 1 toward agent 2 whether agent 1 receives X or Y. If $q_0 = q_{|M|-1}$, the same holds for $(X,Y) \coloneqq (\emptyset, M)$ with $R \coloneqq \{g_1\}$ or $R \coloneqq \{g_{|M|}\}$. So the algorithm asks q_0 and $q_{|M|-1}$ first. If they differ, it keeps indices $a < b$ with $q_a \neq q_b$, initially $a \coloneqq 0$ and $b \coloneqq |M|-1$, and, while $b - a \geq 2$, asks q_c for $c \coloneqq \lfloor (a+b)/2 \rfloor$ and replaces a or b by c so that $q_a \neq q_b$ still

holds; the answers need not be monotone in k. This takes at most $\lceil \log_2 (|M| - 1) \rceil$ queries and ends with $b = a + 1$, so that $k \coloneqq b$ satisfies $q_{k-1} \neq q_k$. Since $Y \neq \emptyset$, one more query on any $g \in Y$ tells whether $v_2 (Y \setminus \{g\}) \leq v_2 (X)$; if so, let $A \coloneqq (Y, X)$, and otherwise, let $A \coloneqq (X, Y)$. Then $R \coloneqq \{g\}$ satisfies the condition for agent 2 toward agent 1, and the condition holds trivially for the pairs (i, i), so $A \in \mathrm{EF1}(\mathcal{I})$, and $\mathrm{SR}_{\mathrm{det}}(\mathrm{EF1}) \leq \lceil \log_2 (m - 1) \rceil + 3$.

Next, suppose that $\mathcal{C}$ is doubly monotone, and let $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$ be the unknown instance in $\mathcal{C}$. The sets G_i in Eqs. (8) and (9) depend only on N and M, so the algorithm fixes them in advance without queries. If $\mathcal{C}$ instead consists of additive instances, the sets $G_i \coloneqq \{g \in M \mid v_i (\emptyset) \leq v_i (\{g\})\}$ satisfy Eqs. (8) and (9) for $\mathcal{I}$, and $|N||M| \leq nm$ queries find them; everything below uses only these conditions on $\mathcal{I}$, and these queries are $O(nm \log \min \{n, m\})$, as $\min \{n, m\} \geq 2$.

We show that an allocation $A \in \mathrm{EF1}(\mathcal{I})$ is computed using $O(nm \log \min \{n, m\})$ queries by Algorithm 4 with no EFL allocation sought, which relies on the subroutines defined in Algorithm 5 and emulates [30, Algorithm 3] as follows:

***Goods phase*:** Algorithm 4 scans all items in $\bigcup_{i \in N} G_i$, which at least one agent considers goods. Every time the current allocation $(A_i)_{i \in N}$ is updated, Line 11 recomputes the set of edges in the *envy graph*. Lines 14 to 18 keep resolving *envy cycles* until the envy graph becomes acyclic, where the $\ell - 1$ transpositions at Line 18 give each agent i_k on the cycle the bundle $A_{i_{k+1}}$. When there is no envy cycle, an item $g \in \bigcup_{i \in N} G_i$ is assigned to an agent corresponding to a source node in the envy graph restricted to the agents for whom g is a good (Line 29).

***Chores phase*:** Algorithm 4 scans all remaining items in $M \setminus \bigcup_{i \in N} G_i$, which all agents consider chores. In addition to the envy graph, the algorithm also considers the *top-trading envy graph*, whose edge set Line 13 computes. Similarly to the goods phase, Lines 14 to 18 keep resolving *top-trading envy cycles* until the top-trading envy graph becomes acyclic. When there is no top-trading envy cycle, an item $g \in M \setminus \bigcup_{i \in N} G_i$ is assigned to an agent corresponding to a sink node in the envy graph (Line 22).

Each resolution of a cycle permutes the bundles so that every agent on the cycle receives a bundle that she strictly prefers, which decreases her out-degree in the envy graph and keeps those of the others, so the algorithm terminates. Under the assumption that Lines 13 and 11 correctly compute the top-trading envy graph and the envy graph, respectively, the proof of [30, Theorem 4] guarantees that, at Line 34, for any $i, j \in N$, $v_i (A_i) \geq v_i (A_j)$ or some $g \in A_i \cup A_j$ satisfies $v_i (A_i \setminus \{g\}) \geq v_i (A_j \setminus \{g\})$, that is, $A \in \mathrm{EF1}(\mathcal{I})$: each step of the goods phase preserves EF1 by their Lemma 5, and each step of the chores phase by their Lemmas 8 and 9; the source at Line 29 exists as the envy graph is then acyclic, and the sink at Line 22 by their Lemma 6, as the top-trading envy graph is then acyclic. Therefore, it remains to prove the correctness of those computations and analyze the overall query complexity.

For each agent $i \in N$, Algorithm 4 maintains as $\mathcal{C}_i$ a clustering of all agents' bundles classified and sorted by the valuation function v_i; more precisely, at the beginning of each iteration over Lines 10 to 33, $\mathcal{C}_i (1), \mathcal{C}_i (2), \ldots$ are nonempty sets that form a partition of $\{(j, A_j) \mid j \in N\}$ and satisfy the following:

- For each $\mathcal{C}_i (k)$, $\{v_i (A_j) \mid (j, A_j) \in \mathcal{C}_i (k)\}$ is a singleton.
- For each pair $\mathcal{C}_i (k_1)$ and $\mathcal{C}_i (k_2)$ with $k_1 < k_2$, any pair $j_1, j_2 \in N$ such that $(j_1, A_{j_1}) \in \mathcal{C}_i (k_1)$ and $(j_2, A_{j_2}) \in \mathcal{C}_i (k_2)$ satisfies that $v_i (A_{j_1}) < v_i (A_{j_2})$.

These invariants are maintained by Lines 16 to 18 and 30 to 32, since Swap in Algorithm 5 exchanges the owners of two bundles and Update $(i, i^*, A_{i^*}, A_{i^*} \cup \{g^*\}; \mathcal{C}_i)$ moves the bundle of i^*

to its place with respect to v_i, thereby verifying the above correctness assumption on Lines 13 and 11 as well.

Algorithm 4 An algorithm that computes an EF1 allocation for a doubly monotone or additive instance, and an EFL allocation for a monotone instance.

Input: The sets of agents N and items M of an unknown instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$ in a set $\mathcal{C}$ of instances that is doubly monotone or consists of additive instances, and whether an EFL allocation is sought, in which case $\mathcal{C}$ consists of monotone instances.

Output: An allocation $A \in \mathrm{EF1}(\mathcal{I})$, and $A \in \mathrm{EFL}(\mathcal{I})$ if an EFL allocation is sought.

1: **for all** $i \in N$ **do**
2: **if** an EFL allocation is sought **then**
3: $G_i \leftarrow M$.
4: **else if** $\mathcal{C}$ is doubly monotone **then**
5: Let $G_i \subseteq M$ be one that satisfies Eqs. (8) and (9) for every instance in $\mathcal{C}$ with agents N and items M.
6: **else**
7: $G_i \leftarrow \{g \in M \mid v_i(\emptyset) \leq v_i(\{g\})\}$, using $|M|$ comparison queries.
8: $\mathcal{C}_i \leftarrow (\{(i', \emptyset) \mid i' \in N\})$.
9: $A_i \leftarrow \emptyset$.
10: **while** $M \neq \emptyset$ **do**
11: $E \leftarrow E' \leftarrow \left\{(i,j) \in N^2 \mid j \in \textsc{GetSuper}(i, A_i; \mathcal{C}_i)\right\}$.
12: **if** $M \cap \bigcup_{i \in N} G_i = \emptyset$ **then**
13: $E' \leftarrow \left\{(i,j) \in N^2 \mid j \in \textsc{GetTop}(i, A_i; \mathcal{C}_i)\right\}$.
14: **if** there exist $\ell \in \{2, 3, \ldots\}$ and distinct $i_1, i_2, \ldots, i_\ell \in N$ such that $(i_k, i_{k+1}) \in E'$ for each $k \in \{1, 2, \ldots, \ell\}$, where $i_{\ell+1} \coloneqq i_1$ **then**
15: **for** $k \leftarrow 1, 2, \ldots, \ell - 1$ **do**
16: **for all** $i \in N$ **do**
17: $\mathcal{C}_i \leftarrow \textsc{Swap}(i_k, A_{i_k}, i_{k+1}, A_{i_{k+1}}; \mathcal{C}_i)$
18: $\left(A_{i_k}, A_{i_{k+1}}\right) \leftarrow \left(A_{i_{k+1}}, A_{i_k}\right)$
19: **else**
20: **if** $M \cap \bigcup_{i \in N} G_i = \emptyset$ **then**
21: Find any $g^* \in M$.
22: Find any $i^* \in N$ such that $\delta^+_{(N,E)}(i^*) = \emptyset$.
23: **else if** an EFL allocation is sought and $A_i = \emptyset$ for some $i \in N$ **then**
24: Find any $i^* \in N$ such that $A_{i^*} = \emptyset$.
25: Find any $g^* \in \arg\max_{g \in M} v_{i^*}(\{g\})$.
26: **else**
27: Find any $g^* \in M \cap \bigcup_{i \in N} G_i$.
28: $N' \leftarrow \{i \in N \mid g^* \in G_i\}$.
29: Find any $i^* \in N'$ such that $\delta^-_{(N', E \cap (N' \times N'))}(i^*) = \emptyset$.
30: **for all** $i \in N$ **do**
31: $\mathcal{C}_i \leftarrow \textsc{Update}(i, i^*, A_{i^*}, A_{i^*} \cup \{g^*\}; \mathcal{C}_i)$
32: $A_{i^*} \leftarrow A_{i^*} \cup \{g^*\}$.
33: $M \leftarrow M \setminus \{g^*\}$.
34: **return** $(A_i)_{i \in N}$.

Apart from the $|N||M|$ queries that find the sets G_i on additive instances, only Lines 30

Algorithm 5 Subroutines for updating the bundle clusters in Algorithm 4.

function UPDATE$(i, j, B, B'; \mathcal{C})$
  Let ℓ be the length of $\mathcal{C}$.
  for all $k \in \{1, 2, \ldots, \ell\}$ **do**
    $\mathcal{C}(k) \leftarrow \mathcal{C}(k) \setminus \{(j, B)\}$.
  Delete the empty set from the sequence $\mathcal{C}$, if any, and let ℓ be the new length of $\mathcal{C}$.
  Find the largest $k^* \in \{0, 1, \ldots, \ell\}$ such that $k^* = 0$ or $v_i(S) \leq v_i(B')$ for each $(j', S) \in \mathcal{C}(k^*)$ using binary search and comparison queries.
  if $k^* > 0$ and a comparison query verifies that $v_i(B') \leq v_i(S)$, hence $v_i(B') = v_i(S)$, with any $(j', S) \in \mathcal{C}(k^*)$ **then**
    $\mathcal{C}(k^*) \leftarrow \mathcal{C}(k^*) \cup \{(j, B')\}$.
  else
    $\mathcal{C} \leftarrow (\mathcal{C}(1), \ldots, \mathcal{C}(k^*), \{(j, B')\}, \mathcal{C}(k^* + 1), \ldots, \mathcal{C}(\ell))$.
  return $\mathcal{C}$.
function SWAP$(i_1, B_1, i_2, B_2; \mathcal{C})$
  Let ℓ be the length of $\mathcal{C}$.
  Find $k_1, k_2 \in \{1, 2, \ldots, \ell\}$ such that $(i_1, B_1) \in \mathcal{C}(k_1)$ and $(i_2, B_2) \in \mathcal{C}(k_2)$.
  $\mathcal{C}(k_1) \leftarrow \mathcal{C}(k_1) \setminus \{(i_1, B_1)\}$ and $\mathcal{C}(k_2) \leftarrow \mathcal{C}(k_2) \setminus \{(i_2, B_2)\}$.
  $\mathcal{C}(k_1) \leftarrow \mathcal{C}(k_1) \cup \{(i_2, B_1)\}$ and $\mathcal{C}(k_2) \leftarrow \mathcal{C}(k_2) \cup \{(i_1, B_2)\}$.
  return $\mathcal{C}$.
function GETTOP$(i, B; \mathcal{C})$
  Let ℓ be the length of $\mathcal{C}$.
  Find $k^* \in \{1, 2, \ldots, \ell\}$ such that $(i, B) \in \mathcal{C}(k^*)$.
  if $k^* = \ell$ **then**
    return $\emptyset$.
  return $\{j \mid (j, S) \in \mathcal{C}(\ell)\}$.
function GETSUPER$(i, B; \mathcal{C})$
  Let ℓ be the length of $\mathcal{C}$.
  Find $k^* \in \{1, 2, \ldots, \ell\}$ such that $(i, B) \in \mathcal{C}(k^*)$.
  return $\left\{j \mid (j, S) \in \bigcup_{k=k^*+1}^{\ell} \mathcal{C}(k)\right\}$.

to 31 make queries, and they run UPDATE $|N||M| \leq nm$ times in total. After its deletion step, each run of UPDATE faces at most $\min\{|N| - 1, m\}$ clusters, since all empty bundles share a cluster and at most $m - 1$ bundles are nonempty while g^* is unallocated, so it makes at most $\lceil \log_2(\min\{n - 1, m\} + 1) \rceil + 1$ queries. This gives $O(nm \log \min\{n, m\})$ queries in total, as $\min\{n, m\} \geq 2$.

Lastly, suppose that $\mathcal{C}$ consists of monotone instances, and let $\mathcal{I} \in \mathcal{C}$ be the unknown instance, where $\mathcal{I} = (N, M, (v_i)_{i \in N})$. We show that an allocation $A \in \mathrm{EFL}(\mathcal{I})$ is computed using $O(nm \log \min\{n, m\})$ queries by Algorithm 4 with an EFL allocation sought, so that $G_i = M$ for every agent i; this emulates the algorithm implied in [65, Proof of Lemma 18], which is Akrami et al. [6, Lemma 3.8]. As before, the clusterings give the envy graph at Line 11 without queries, the top-trading envy graph is never used, and the algorithm terminates. By monotonicity, no agent envies an empty bundle, so an agent with an empty bundle lies on no envy cycle, and her first nonempty bundle is a single item g that maximizes $v_i(\{g\})$ over the remaining items (Lines 23 to 25). Moreover, the value $v_i(A_i)$ of each agent i never decreases, and an item is added to a nonempty bundle only when every bundle is nonempty and its holder is a source of the envy graph.

Hence, if $|A_j| \geq 2$ at Line 34 and g is the item last added to the bundle A_j, then every agent i satisfies $v_i(A_i) \geq v_i(A_j \setminus \{g\})$, as she did not envy the holder of $A_j \setminus \{g\}$ at that time, or, if she is this holder, by monotonicity, and $v_i(A_i) \geq v_i(\{g\})$, as g remained when she chose her first item. Therefore, $(A_i)_{i \in N} \in \mathrm{EFL}(\mathcal{I})$ at Line 34. Besides the $|N||M|$ runs of Update, which make $O(nm \log \min\{n, m\})$ queries as above, Line 25 makes at most $|M| - 1$ queries each time and runs at most $|N|$ times, since the number of empty bundles never increases. This adds at most nm queries, which is $O(nm \log \min\{n, m\})$ as $\min\{n, m\} \geq 2$. □

Algorithm 6 An algorithm for EF1 verification.

Input: The sets of agents N and items M of an unknown instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$ and an allocation $A \in \mathcal{A}(\mathcal{I})$.
Output: Whether $A \in \mathrm{EF1}(\mathcal{I})$ or not.

1: $F \leftarrow \left\{(i, j) \in N^2 \mid i = j \vee A_i = A_j = \emptyset\right\}$.
2: **for all** $(i, j) \in N^2$ such that $i \neq j$ and $A_i \cup A_j \neq \emptyset$ **do**
3: **if** a comparison query verifies that $v_i(A_i) \geq v_i(A_j)$ **then**
4: $F \leftarrow F \cup \{(i, j)\}$.
5: **for all** $i \in N$ **do**
6: **for all** $g \in A_i$ **do**
7: **for all** $j \in N \setminus \{i\}$ **do**
8: **if** a comparison query verifies that $v_i(A_i \setminus \{g\}) \geq v_i(A_j)$ **then**
9: $F \leftarrow F \cup \{(i, j)\}$.
10: **if** a comparison query verifies that $v_j(A_i \setminus \{g\}) \leq v_j(A_j)$ **then**
11: $F \leftarrow F \cup \{(j, i)\}$.
12: **return** whether $(F = N^2 \wedge A \in \mathcal{A}_0(\mathcal{I}))$ or not.

Algorithm 7 An algorithm for EFL verification.

Input: The sets of agents N and items M of an unknown instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$ and an allocation $A \in \mathcal{A}(\mathcal{I})$.
Output: Whether $A \in \mathrm{EFL}(\mathcal{I})$ or not.

1: $F \leftarrow \left\{(i, j) \in N^2 \mid |A_j| \leq 1\right\}$.
2: **for all** $j \in N \mid |A_j| > 1$ **do**
3: **for all** $g \in A_j$ **do**
4: **for all** $i \in N$ **do**
5: **if** two comparison queries verify that $v_i(A_i) \geq \max\{v_i(\{g\}), v_i(A_j \setminus \{g\})\}$ **then**
6: $F \leftarrow F \cup \{(i, j)\}$.
7: **return** whether $(F = N^2 \wedge A \in \mathcal{A}_0(\mathcal{I}))$ or not.

Proof of Proposition 2. Let $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$ be the unknown instance, $A \in \mathcal{A}(\mathcal{I})$ be the given allocation, and $c \in \{0, 1, \ldots\}$ be arbitrary. The algorithms decide whether $A \in \mathcal{A}_0(\mathcal{I})$ or $A \in \mathcal{A}_c(\mathcal{I})$ without queries, and each comparison query decides an inequality and thus also its strict negation.

For EF1, every pair (i, i) satisfies the condition in Definition 6 with $R = \emptyset$, so fix any $i, j \in N$ with $i \neq j$, whose bundles A_i and A_j are disjoint. By Definition 6, the pair (i, j) satisfies the condition with $R = \{g\}$ for $g \in A_i$ if and only if $v_i(A_i \setminus \{g\}) \geq v_i(A_j)$, with $R = \{g\}$ for $g \in A_j$ if and only if $v_i(A_i) \geq v_i(A_j \setminus \{g\})$, and with $R = \emptyset$, or $R = \{g\}$ for $g \notin A_i \cup A_j$, if and only if

$v_i(A_i) \geq v_i(A_j)$. In particular, it does so if $A_i = A_j = \emptyset$. Algorithm 6 puts these pairs into F at the start, tests the third case by one query for each other pair, and, for each $i \in N$, $g \in A_i$, and $j \in N \setminus \{i\}$, tests the first case for (i, j) and the second case for (j, i) by one query each. Hence, $F = N^2$ at the end if and only if every pair satisfies the condition, so Algorithm 6 is correct. It makes at most $2|N||M|$ queries in the last loop, as $\sum_{i \in N} |A_i| \leq |M|$, and at most $2|N||M|$ before it, as each pair tested there contains one of the at most $|M|$ agents with nonempty bundles, each of which is in at most $2|N|$ pairs.

For EFL, Definition 6 imposes no condition on the pairs (i, j) with $|A_j| \leq 1$, which Algorithm 7 puts into F at the start, and, for each other pair, it tests every $g \in A_j$ by at most two queries, which is correct and makes at most $2|N||M|$ queries.

Algorithm 8 An algorithm for EFX verification.

Input: The sets of agents N and items M of an unknown instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$, an allocation $A \in \mathcal{A}(\mathcal{I})$, and $c \in \{0, 1, \ldots\}$.

Output: Whether $A \in \mathrm{EFX}_c(\mathcal{I})$ or not.

```
for all i ∈ N do
    for all g ∈ A_i do
        for all j ∈ N \ {i} do
            if two comparison queries together verify that v_i(A_i) < v_i(A_i \ {g}) < v_i(A_j) then
                return False.
            if two comparison queries together verify that v_j(A_i) > v_j(A_i \ {g}) > v_j(A_j) then
                return False.
return whether A ∈ A_c(I) or not.
```

Algorithm 9 An algorithm for EFX^+ verification.

Input: The sets of agents N and items M of an unknown instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$, an allocation $A \in \mathcal{A}(\mathcal{I})$, and $c \in \{0, 1, \ldots\}$.

Output: Whether $A \in \mathrm{EFX}_c^+(\mathcal{I})$ or not.

```
for all i ∈ N do
    for all g ∈ A_i do
        for all j ∈ N \ {i} do
            if two comparison queries together verify that v_i(A_i) < v_i(A_i \ {g}) < v_i(A_j) then
                return False.
            if two comparison queries together verify that v_j(A_i) ≥ v_j(A_i \ {g}) > v_j(A_j) then
                return False.
return whether A ∈ A_c(I) or not.
```

For EFX_c, since $M_{j,i}^+ \subseteq A_i$ and $M_i^- \subseteq A_i$, every inequality in Definition 7 is indexed by some $i \in N$, $g \in A_i$, and $j \in N$, and it is violated if and only if $v_j(A_i) > v_j(A_i \setminus \{g\}) > v_j(A_j)$ for the first family and $v_i(A_i) < v_i(A_i \setminus \{g\}) < v_i(A_j)$ for the second, neither of which can hold if $j = i$. Algorithm 8 tests both by at most four queries for each such (i, g, j) with $j \neq i$ and then checks whether $A \in \mathcal{A}_c(\mathcal{I})$, which is correct and makes at most $4|N||M|$ queries. The same holds for EFX_c^+ and Algorithm 9, where the first violation reads $v_j(A_i) \geq v_j(A_i \setminus \{g\}) > v_j(A_j)$. As $|N| \leq n$ and $|M| \leq m$, we obtain $\mathrm{VR}_{\mathrm{det}}(\mathcal{P}) \leq 4nm$ for each $\mathcal{P} \in \{\mathrm{EF1}, \mathrm{EFL}\} \cup \left\{\mathrm{EFX}_c, \mathrm{EFX}_c^+ \mid c \in \{0, 1, \ldots\}\right\}$.

Lastly, suppose that $\mathcal{C}$ consists of monotone additive instances. Then $M_i^- = \emptyset$, and, with equality allowed, $M_{i,j}^+ = A_j$ for all $i, j \in N$, so $A \in \mathrm{EFX}_0^+(\mathcal{I})$ if and only if $A \in \mathcal{A}_0(\mathcal{I})$ and $v_i(A_i) \geq$

Algorithm 10 An algorithm for EFX_0^+ verification in monotone additive instances.

Input: The sets of agents N and items M of an unknown monotone additive instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$ and an allocation $A \in \mathcal{A}(\mathcal{I})$.
Output: Whether $A \in \mathrm{EFX}_0^+(\mathcal{I})$ or not.

```
if A ∉ A_0(I) then
    return False.
U ← ⋃{A_j | j ∈ N, |A_j| ≥ 2}.
for all i ∈ N do
    if A_i ≠ ∅ then
        for all j ∈ N \ {i} with |A_j| ≥ 2 do
            for all g ∈ A_j do
                if a comparison query verifies that v_i(A_j \ {g}) > v_i(A_i) then
                    return False.
    else if U ≠ ∅ and a comparison query verifies that v_i(U) > v_i(∅) then
        return False.
return True.
```

$v_i(A_j \setminus \{g\})$ for all $i, j \in N$ and $g \in A_j$, which holds for $j = i$ by monotonicity. Algorithm 10 tests these inequalities directly for each agent i with $A_i \neq \emptyset$ and each $j \neq i$ with $|A_j| \geq 2$, as they hold by monotonicity if $|A_j| = 1$, using at most $|M|$ queries. For an agent i with $A_i = \emptyset$, they hold trivially if $|A_j| \leq 1$, and, if $|A_j| \geq 2$, they hold for all $g \in A_j$ if and only if $v_i(\{h\}) = 0$ for every $h \in A_j$, since v_i is additive with nonnegative item values and every $h \in A_j$ lies in $A_j \setminus \{g\}$ for some $g \in A_j$. Thus they hold for all $j \in N$ if and only if $v_i(U) \leq v_i(\emptyset)$ for the set U in Algorithm 10, which one query tests. Since at most $\min\{|N|, |M|\}$ agents have nonempty bundles, Algorithm 10 is correct and makes at most $m \min\{n, m\} + n$ queries.

Finally, suppose that $\mathcal{C}$ consists of antitone additive instances, so that $M_{i,j}^+ = \emptyset$ for all $i, j \in N$, and $A \in \mathrm{EFX}_0(\mathcal{I})$ if and only if A is complete and $v_i(A_i \setminus \{g\}) \geq v_i(A_j)$ for all $i, j \in N$ and $g \in M_i^-$, which holds for $j = i$. For each $i \in N$, these inequalities hold if $M_i^- = \emptyset$, and, otherwise, if and only if $v_i(A_i \setminus \{g_i\}) \geq v_i(A_{j_i})$ for some $g_i \in M_i^-$ of largest value $v_i(\{g_i\})$ and some $j_i \in N \setminus \{i\}$ of largest value $v_i(A_{j_i})$. For each agent i with $A_i \neq \emptyset$, the $|A_i|$ comparisons of $\emptyset$ with each $\{g\}$ for $g \in A_i$ give $M_i^- = \{g \in A_i \mid v_i(\{g\}) < 0\}$, at most $|A_i| - 1$ more give g_i, at most $\max\{|N| - 2, 0\}$ give j_i, and one more tests the inequality, which makes at most $2|M| + |N| \max\{|N| - 2, 0\} \leq 2m + n^2$ comparisons in total. If, moreover, $\mathcal{C}$ consists of instances with n agents and m items, fewer comparisons suffice. If some bundle of A is empty, it is a best other bundle for every agent i with $A_i \neq \emptyset$, and, as item values are nonpositive, $v_i(A_i \setminus \{g\}) \geq 0$ for every $g \in M_i^-$ if and only if $\left|M_i^-\right| \leq 1$; so the comparisons that give the sets M_i^- suffice, at most m in total. Otherwise, every bundle is nonempty, every agent i with $|A_i| = 1$ satisfies the inequalities, as $v_i(A_i \setminus \{g\}) = 0 \geq v_i(A_j)$, and at most $\min\{n, m - n\}$ agents have $|A_i| \geq 2$, as the n nonempty bundles share m items; checking only these agents makes at most $2m + (n - 2)\min\{n, m - n\}$ comparisons. □

Proof of Theorem 16. First, suppose that $\mathcal{C}$ consists of additive instances with two agents. As noted at the beginning of Appendix B, we may assume without loss of generality that $\mathcal{C}$ contains every such instance with at most m items. Let $\mathcal{I} \in \mathcal{C}$ be the unknown instance, where $\mathcal{I} = (\{1, 2\}, M, (v_1, v_2))$, and define $\mathcal{I}' \coloneqq (\{1, 2\}, M, (v_1, v_1)) \in \mathcal{C}$. Due to Theorem 32, one can find $A' \in \mathrm{EFX}_0^+(\mathcal{I}') \cap \mathrm{PO}(\mathcal{I}')$ using $O(|M| \log |M|)$ queries, which are comparisons for v_1 that agent 1 answers. It then follows

that $A \in \mathrm{EFX}_0^+(\mathcal{I})$, with $A \in \mathcal{A}_0(\mathcal{I})$ defined as follows:

$$(A_1, A_2) \coloneqq \begin{cases} (A_1', A_2') & \text{if } v_2(A_1') < v_2(A_2'); \\ (A_2', A_1') & \text{otherwise.} \end{cases} \tag{26}$$

Indeed, agent 1 satisfies the conditions, as $\mathcal{I}'$ is homogeneous and $A' \in \mathrm{EFX}_0^+(\mathcal{I}')$, and agent 2 satisfies them because $v_2(A_2) \geq v_2(A_1)$, removing an item of nonnegative value does not increase $v_2(A_1)$, and removing an item of negative value increases $v_2(A_2)$, as v_2 is additive. We thus establish that $\mathrm{SR}_{\det}(\mathrm{EFX}_0^+) = O(m \log m)$, which implies the same bound for EFX_0.

Next, suppose that $\mathcal{C}$ consists of monotone additive instances with n agents and m items; as noted at the beginning of Appendix B, we may assume that it contains every such instance, so that it is of the form required by Theorem 1, with the nonnegative orthant as every domain. If $n \in \{3, 4\}$, then $\mathrm{EFX}_{n-3}^+(\mathcal{I}) \neq \emptyset$ for every $\mathcal{I} \in \mathcal{C}$ [44, 29], and $\mathrm{VR}_{\det}(\mathrm{EFX}_{n-3}^+) = O(m)$ by Proposition 2, so Corollary 1 gives $\mathrm{SR}_{\det}(\mathrm{EFX}_{n-3}^+) = O\left(m \cdot m^2 / \log m\right)$. For every n, $\mathrm{EFX}_{n-2}^+(\mathcal{I}) \neq \emptyset$ for every $\mathcal{I} \in \mathcal{C}$ [29], and $\mathrm{VR}_{\det}(\mathrm{EFX}_{n-2}^+) = O(nm)$, so Corollary 1 gives $\mathrm{SR}_{\det}(\mathrm{EFX}_{n-2}^+) = O\left(nm \cdot nm^2 / \log(nm)\right)$, as $x / \log x$ increases for $x \geq e$. □

Algorithm 11 An algorithm for XEF1 verification.

Input: The sets of agents N and items M of an unknown instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$ and an allocation $A \in \mathcal{A}(\mathcal{I})$.

Output: Whether $A \in \mathrm{XEF1}(\mathcal{I})$ or not.

```
for all i ∈ N do
    S_i ← ∅.
    for all B ∈ { A_j △ R | (j, R) ∈ N × (M choose ≤1) } do
        if a comparison query verifies that v_i(A_i) ≥ v_i(B) then
            S_i ← S_i ∪ {B}.
    for all j ∈ N do
        if S_i ∩ { A_j △ R | R ∈ (M choose ≤1) } = ∅ then
            return False.
return whether A ∈ A_0(I) or not.
```

Proof of Proposition 3. Let $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$ be the unknown instance and $A \in \mathcal{A}(\mathcal{I})$ be the given allocation. The algorithms decide whether $A \in \mathcal{A}_0(\mathcal{I})$ without queries, and every pair (i, i) satisfies the conditions in Definition 8 with $R = \emptyset$. At most $\min\{|N|, |M| + 1\}$ of the bundles A_j are distinct, as they are pairwise disjoint, so that at most $|M|$ of them are nonempty.

For IEF1, the bundle B_i of Algorithm 12 maximizes $v_i(A_i \triangle R)$ over $R \in \binom{M}{\leq 1}$, so a pair (i, j) satisfies the condition if and only if $v_i(B_i) \geq v_i(A_j)$, which holds if $A_j = A_i$. Hence Algorithm 12 is correct, and it makes at most $|M| + \min\{|N|, |M| + 1\} - 1 \leq 2|M|$ queries for each agent, so $\mathrm{VR}_{\det}(\mathrm{IEF1}) \leq 2nm$.

In general, $O(|N||M| \min\{|N|, |M|\})$ queries suffice for Algorithm 11, which tests the condition of XEF1 directly and is thus correct. If $\mathcal{C}$ consists of monotone instances, then $\mathrm{XEF1}(\mathcal{I}) = \mathrm{EF1}(\mathcal{I})$, so Proposition 2 gives $\mathrm{VR}_{\det}(\mathrm{XEF1}) = O(nm)$. Indeed, for $i \neq j$ and monotone v_i, the condition of XEF1 with $R = \{g\}$ for $g \notin A_j$ implies $v_i(A_i) \geq v_i(A_j \cup \{g\}) \geq v_i(A_j)$, and that of EF1 with $R = \{g\}$ for $g \in A_i$ implies $v_i(A_i) \geq v_i(A_i \setminus \{g\}) \geq v_i(A_j)$, so both conditions hold if and only if $v_i(A_i) \geq v_i(A_j \setminus R)$ for some $R \in \binom{A_j}{\leq 1}$. Symmetrically, $\mathrm{IEF1}(\mathcal{I}) = \mathrm{EF1}(\mathcal{I})$ if $\mathcal{I}$ is antitone, as both conditions then hold if and only if $v_i(A_i \setminus R) \geq v_i(A_j)$ for some $R \in \binom{A_i}{\leq 1}$.

Algorithm 12 An algorithm for IEF1 verification.

Input: The sets of agents N and items M of an unknown instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$ and an allocation $A \in \mathcal{A}(\mathcal{I})$.

Output: Whether $A \in \mathrm{IEF1}(\mathcal{I})$ or not.

if $A \notin \mathcal{A}_0(\mathcal{I})$ **then**
  return `False`.
for all $i \in N$ **do**
  Find $B_i \in \left\{A_i \triangle R \mid R \in \binom{M}{\leq 1}\right\}$ that maximizes v_i, using $|M|$ comparison queries.
  for all $B \in \{A_j \mid j \in N\}$ such that $B \neq A_i$ **do**
    if a comparison query verifies that $v_i(B_i) < v_i(B)$ **then**
      return `False`.
return `True`.

Lastly, suppose that $\mathcal{C}$ consists of additive instances, and reject A without queries if $A \notin \mathcal{A}_0(\mathcal{I})$. For each $i \in N$ and $B \subseteq M$, $v_i(B \triangle \{g\})$ equals $v_i(B) - v_i(\{g\})$ for $g \in B$ and $v_i(B) + v_i(\{g\})$ for $g \notin B$, so $v_i(B \triangle R)$ is minimized over $R \in \binom{M}{\leq 1}$ by $R = \emptyset$, by $R = \{g\}$ for some $g \in B$ maximizing $v_i(\{g\})$, or by $R = \{h\}$ for some $h \in M \setminus B$ minimizing $v_i(\{h\})$. For each $i \in N$, at most $2|M|$ queries find an item of maximum and one of minimum value $v_i(\{g\})$ in each nonempty bundle A_j, and at most $2|M|$ more find, among these minima, two of the smallest ones that lie in distinct bundles; since A is complete, the first of them is an item h of minimum value outside every bundle $B = A_j$ that does not contain it, and the second one outside the bundle that does. Then the pair (i, j) satisfies the condition if and only if one of at most three queries verifies that $v_i(A_i) \geq v_i(B)$, $v_i(A_i) \geq v_i(B \setminus \{g\})$, or $v_i(A_i) \geq v_i(B \cup \{h\})$ for $B = A_j$ and these items g and h, where existent, and these queries are made once for each of the at most $\min\{|N|, |M|+1\}$ distinct bundles. This makes $O(|M|)$ queries for each agent, so $\mathrm{VR}_{\mathrm{det}}(\mathrm{XEF1}) = O(nm)$. □

Proof of Theorem 17. Let $N \coloneqq \{1, 2, \ldots, n\}$ and $M \coloneqq \{1, 2, \ldots, m\}$, and suppose first that $m \geq n+1$, so that $1 \leq k \leq n-1$. Let $M^\star \coloneqq \{1, 2, \ldots, n+k\} \subseteq M$, and let $B \in \Pi_n(M)$ be given by $B_i \coloneqq M \cap \{i + qn \mid q \in \{0, 1, \ldots\}\}$ for each $i \in N$, so that $B_i \cap M^\star = \{i, i+n\}$ if $i \leq k$ and $B_i \cap M^\star = \{i\}$ otherwise. Let $L \coloneqq k(n-k)$, which is at least $n \min\{n, m-n\}/6$, as $n - k \geq \lfloor n/2 \rfloor \geq n/3$ and $k \geq \min\{n, m-n\}/2$.

We first allow value queries. Let $\mathcal{J}$ be the instance with agents N, items M, and the additive valuations $u_i(S) \coloneqq |S \cap M^\star|$ for $S \subseteq M$ and $i \in N$, which are monotone. At $\mathcal{J}$, a complete allocation A satisfies the condition of EF1 for agents i and j if and only if $|A_i \cap M^\star| \geq |A_j \cap M^\star| - 1$, as removing an item of $A_j \cap M^\star$, if any, is best. So the numbers $|A_i \cap M^\star|$ for $A \in \mathrm{EF1}(\mathcal{J})$ differ by at most 1 and sum to $n+k$, where $n < n+k < 2n$; hence A gives one item of $M^\star$ to each agent of a set $N_1(A)$ of $n-k$ agents and two to each other agent, and, conversely, every such complete allocation, including B, lies in $\mathrm{EF1}(\mathcal{J})$. We claim that, for every $A \in \mathrm{EF1}(\mathcal{J})$ and every set Q of fewer than L queries, some monotone additive instance $\mathcal{J}'$ with agents N and items M satisfies $Q \cap \Delta(\mathcal{J}, \mathcal{J}') = \emptyset$ and $A \notin \mathrm{EF1}(\mathcal{J}')$. As in the proof of Theorem 10, Lemmas 11 and 16 then give $\mathrm{SR}_{\mathrm{rand}}(\mathrm{EF1}) \geq L$ and $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EF1}) \geq L$, even if value queries are allowed.

To prove the claim, note that, as $|Q| < k(n-k)$, some $i^* \in N_1(A)$ receives fewer than k of the queries in Q; let a be the item of $A_{i^*} \cap M^\star$, and let $H \subseteq \mathbb{R}^{M^\star}$ be the span of the restrictions to $M^\star$ of the vectors of these queries, as defined in the proof of Theorem 10. For $j \in N \setminus N_1(A)$, let W_j be the span of the two vectors $\mathbf{1}_{M^\star \setminus \{g\}} - \mathbf{1}_{M^\star \setminus \{a\}}$ with $g \in A_j \cap M^\star$; these $2k$ vectors are linearly independent, so the sum of the k spaces W_j is direct. If each W_j contained a nonzero vector of H, these k vectors would be linearly independent, although H has dimension less than k;

so some j^* satisfies $W_{j^*} \cap H = \{\mathbf{0}\}$. Then the linear map $d \mapsto (d_g - d_a)_{g \in A_{j^*} \cap M^\star}$ on the orthogonal complement $H^\perp$ of H in $\mathbb{R}^{M^\star}$ is onto $\mathbb{R}^{A_{j^*} \cap M^\star}$, since otherwise some nonzero $w \in W_{j^*}$ would be orthogonal to $H^\perp$ and thus lie in H. Hence some $d \in H^\perp$ satisfies $d_g > d_a$ for both $g \in A_{j^*} \cap M^\star$ and, after scaling, $|d_g| < 1$ for every $g \in M^\star$. Let $\mathcal{J}'$ be obtained from $\mathcal{J}$ by replacing u_{i^*} with the additive function that values each $g \in M^\star$ at $1 + d_g > 0$ and every other item at 0, which is monotone; the two valuations of i^* differ by d extended by zeros, which is orthogonal to the vectors of all queries in Q to i^*, so $Q \cap \Delta(\mathcal{J}, \mathcal{J}') = \emptyset$. But for every $R \in \binom{M}{\leq 1}$, agent i^* values $A_{i^*} \setminus R$ at $\mathcal{J}'$ at most at $1 + d_a$ and $A_{j^*} \setminus R$, which keeps an item of $A_{j^*} \cap M^\star$, above $1 + d_a$, so $A \notin \mathrm{EF1}(\mathcal{J}')$.

Now let only comparison queries be allowed, and let $\varepsilon \in [0, 1/2)$ and $N_1 \coloneqq N_1(B) = \{k + 1, \dots, n\}$. For each $i \in N_1$, fix an injective map $r_i : M^\star \setminus \{i\} \to \{0, 1, \dots, n + k - 2\}$, and, for $g \in M^\star \setminus \{i\}$, let $\sigma_{i,g} \coloneqq 1$ if $g \leq k$ and $\sigma_{i,g} \coloneqq -1$ otherwise, so that the two items j and $j + n$ of $B_j \cap M^\star$ with $j \in N \setminus N_1$ get opposite signs. With $\delta \coloneqq 2^{-n-k}$, let $\mathcal{I}_0$ be the instance with agents N, items M, and the additive valuations $v_i \coloneqq \mathbf{1}_{M|M^\star} + \delta \sum_{g \in M^\star \setminus \{i\}} \sigma_{i,g} 2^{r_i(g)} \mathbf{1}_{M|\{g\}}$ for $i \in N_1$ and $v_i \coloneqq \mathbf{1}_{M|M^\star}$ for $i \in N \setminus N_1$, which are monotone as $\delta 2^{n+k-2} < 1$. Then $B \in \mathrm{EF1}(\mathcal{I}_0)$: each $i \in N \setminus N_1$ values B_i at 2 and no bundle above 2, and each $i \in N_1$ values B_i at 1, every other B_j with $j \in N_1$ below 1, and $B_j \setminus \{j\}$ with $j \in N \setminus N_1$ at $1 - \delta 2^{r_i(j+n)} < 1$. For $i \in N_1$ and $j \in N \setminus N_1$, let $\mathcal{I}_{i,j}$ be obtained from $\mathcal{I}_0$ by flipping the sign $\sigma_{i,j+n}$; then agent i values both items of $B_j \cap M^\star$ above 1, hence $B_j \setminus R$ above 1 and $B_i \setminus R$ at most at 1 for every $R \in \binom{M}{\leq 1}$, so $B \notin \mathrm{EF1}(\mathcal{I}_{i,j})$. Since $\delta < 2^{-1-(n+k-2)}$, Lemma 14 applies to v_i with $c \coloneqq \mathbf{1}_{M|M^\star}$, $U \coloneqq M^\star \setminus \{i\}$, $\sigma \coloneqq (\sigma_{i,g})_{g \in U}$, and $r \coloneqq r_i$, so $\mathcal{I}_{i,j}$ answers a comparison differently from $\mathcal{I}_0$ only if it is a comparison to agent i whose normal h satisfies $t(h) = j + n$. So the L sets $\Delta((\mathcal{I}_0, B), (\mathcal{I}_{i,j}, B))$ are pairwise disjoint, and, as B is valid at $\mathcal{I}_0$ and invalid at every $\mathcal{I}_{i,j}$, Lemma 12 with unit weights shows that every algorithm that verifies correctly with probability at least $1 - \varepsilon$ makes at least $(1 - 2\varepsilon) L$ queries in expectation on $(\mathcal{I}_0, B)$.

For the bound $\Omega(ns \log(1 + m/s))$, still with comparison queries only, let $m \geq 2$ be arbitrary and $p \coloneqq \lfloor s/2 \rfloor \geq 1$, and let $A^\circ \in \Pi_n(M)$ give each agent $j \leq p$ a set A°_j of $b_j \in \{\lfloor m/p \rfloor, \lceil m/p \rceil\}$ items and the other agents nothing, where $b_j \geq 2$ as $m \geq 2p$. For each choice of items $\tau_{i,j} \in A^\circ_j$ with $i \in N \setminus \{1, \dots, p\}$ and $j \in \{1, \dots, p\}$, let $\mathcal{I}_\tau$ be the instance with agents N, items M, and the additive valuations $v_j \coloneqq \mathbf{0}$ for $j \leq p$ and $v_i \coloneqq \sum_{j=1}^{p} \mathbf{1}_{M|\{\tau_{i,j}\}}$ for $i > p$, and let $\mathcal{F}$ be the set of these $\prod_{j=1}^{p} b_j^{n-p}$ monotone additive instances. At every $\mathcal{I}_\tau \in \mathcal{F}$, each agent $i > p$ values $A^\circ_j \setminus \{\tau_{i,j}\}$ at $0 = v_i(A^\circ_i)$ for each $j \leq p$, and the other agents value everything at 0, so $A^\circ \in \mathrm{EF1}(\mathcal{I}_\tau)$. At the midpoint of distinct $\mathcal{I}_\tau, \mathcal{I}_{\tau'} \in \mathcal{F}$, which is monotone additive, some agent $i > p$ values two items $\tau_{i,j} \neq \tau'_{i,j}$ of some A°_j at $1/2$ each, so she values $A^\circ_j \setminus R$ at least at $1/2 > 0 = v_i(A^\circ_i \setminus R)$ for every $R \in \binom{M}{\leq 1}$, and A° is not EF1 there. The same holds for EFX_0, as $\tau_{i,j}$ is the only item of A°_j whose removal lowers the value of agent i at $\mathcal{I}_\tau$, while at the midpoint removing $\tau_{i,j}$ lowers it and leaves at least $1/2$: $A^\circ \in \mathrm{EFX}_0(\mathcal{I})$ for every $\mathcal{I} \in \mathcal{F}$, but $A^\circ \notin \mathrm{EFX}_0\left(\frac{1}{2}(\mathcal{I} + \mathcal{I}')\right)$ for all distinct $\mathcal{I}, \mathcal{I}' \in \mathcal{F}$. So Lemma 15 gives $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EF1}) \geq \log_2 |\mathcal{F}| = (n - p) \sum_{j=1}^{p} \log_2 b_j$, which is at least $ns \log_2(1 + m/s)/12$, as $n - p \geq n/2$, $p \geq s/3$, and $b_j \geq \max\{2, m/(2p)\} \geq \max\{2, m/s\}$, so that $b_j^2 \geq 2m/s \geq 1 + m/s$.

Finally, let $\mathcal{C}$ be the set of all additive instances with n agents and m items, and allow value queries again. Let $A^\star \coloneqq (\emptyset, M, \emptyset, \dots, \emptyset)$, and let $\mathcal{I}'_0$ be the instance with agents N and items M in which agent 1 has the additive valuation $v_1 \coloneqq \mathbf{1}_{M|M}$ and every other agent the valuation $\mathbf{0}$; then $A^\star \notin \mathrm{EF1}(\mathcal{I}'_0)$, as $v_1(M \setminus R) \geq m - 1 > 0 = v_1(\emptyset)$ for every $R \in \binom{M}{\leq 1}$. For a set Q of fewer than m queries, some nonzero $d \in \mathbb{R}^M$ is orthogonal to the vectors of the queries in Q to agent 1, and $\langle d, \mathbf{1}_{M|M\setminus\{g\}} \rangle \neq 0$ for some $g \in M$, since these m vectors span $\mathbb{R}^M$ as $m \geq 2$. Scaling d so that this inner product is $1 - m$, let $\mathcal{I}'_Q$ be obtained from $\mathcal{I}'_0$ by replacing v_1 with the additive function $v_1 + d$;

then $Q \cap \Delta\left(\mathcal{I}'_0, \mathcal{I}'_Q\right) = \emptyset$, as in the proof of Theorem 10, and $A^\star \in \mathrm{EF1}\left(\mathcal{I}'_Q\right)$, as agent 1 values $M \setminus \{g\}$ at 0 and every other agent values every bundle at 0. A deterministic verifier that is correct on $(\mathcal{I}'_0, A^\star)$ and on every such $\left(\mathcal{I}'_Q, A^\star\right)$ rejects $(\mathcal{I}'_0, A^\star)$, and if it made a set Q of fewer than m queries there, it would receive the same answers on $\left(\mathcal{I}'_Q, A^\star\right)$ and wrongly reject it; so Lemma 11 gives $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EF1}) \geq m$, even if value queries are allowed.

Finally, the bounds for monotone additive instances hold for XEF1, as $\mathrm{XEF1}(\mathcal{I}) = \mathrm{EF1}(\mathcal{I})$ for every monotone additive instance $\mathcal{I}$ (proof of Proposition 3). For EFL, let $m \geq n+1$, and let $\mathcal{J}$, B, and L be as above. As $\mathrm{EFL}(\mathcal{I}) \subseteq \mathrm{EF1}(\mathcal{I})$ for every monotone $\mathcal{I}$ (Definition 7), the claim above applies to every $A \in \mathrm{EFL}(\mathcal{J})$, and every $\mathcal{J}'$ that it gives satisfies $A \notin \mathrm{EFL}(\mathcal{J}')$. Moreover, $B \in \mathrm{EFL}(\mathcal{J})$: every agent receives an item of $M^\star$, and, for every $j \in N$ with $|B_j| > 1$, an item $g \in B_j \cap M^\star$ satisfies $u_i(\{g\}) = 1$ and $u_i(B_j \setminus \{g\}) \leq 1$, which are at most $u_i(B_i)$ for every $i \in N$. So Lemma 16, with EF1 in place of $\mathcal{P}$ and EFL in place of $\mathcal{P}'$, gives $\mathrm{SR}_{\mathrm{rand}}(\mathrm{EFL}) \geq L$, and the argument for verification above, with the input $(\mathcal{J}, B)$, gives $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EFL}) \geq L$, even if value queries are allowed. For IEF1, let $m \geq n+1$, and let $\mathcal{J}$, B, and L be as above. At $\mathcal{J}$, toggling one item raises the value of an agent by at most 1, and by exactly 1 if she lacks an item of $M^\star$ and adds one, so a complete allocation satisfies the condition of IEF1 for agents i and j if and only if $|A_i \cap M^\star| \geq |A_j \cap M^\star| - 1$, as for EF1. Hence $\mathrm{IEF1}(\mathcal{J}) = \mathrm{EF1}(\mathcal{J})$, which contains B, and we prove the claim above with IEF1 in place of EF1. Let $A \in \mathrm{IEF1}(\mathcal{J})$ and Q, i^*, a, and H be as there, and, for $j \in N \setminus N_1(A)$, let $D_j \coloneqq A_j \cap M^\star$ and $z_{j,g} \coloneqq \mathbf{1}_{M^\star|D_j} - \mathbf{1}_{M^\star|\{a\}} - \mathbf{1}_{M^\star|\{g\}}$ for $g \in M^\star \setminus \{a\}$. Suppose that, for every such j, no $d \in H^\perp$ satisfies $\langle z_{j,g}, d\rangle > 0$ for all $g \in M^\star \setminus \{a\}$. By Gordan's theorem, applied to the orthogonal projections of these vectors onto $H^\perp$, there are then weights $\lambda_{j,g} \geq 0$ with $L_j \coloneqq \sum_g \lambda_{j,g} > 0$ and $w_j \coloneqq \sum_g \lambda_{j,g} z_{j,g} \in H$. The coordinates of w_j sum to $2L_j - \sum_{g \in D_j} \lambda_{j,g} \geq L_j$ over D_j, and to $-\sum_{g \in D_h} \lambda_{j,g} \leq 0$ over D_h for every other $h \in N \setminus N_1(A)$, as $a \notin D_h$; the absolute values of the latter sum to at most $L_j - \sum_{g \in D_j} \lambda_{j,g}$. So the $k \times k$ matrix of these block sums is strictly diagonally dominant in every column, hence nonsingular, and the k vectors $w_j \in H$ are linearly independent, although H has dimension less than k. Therefore some j^* and some $d \in H^\perp$ satisfy $\langle z_{j^*,g}, d\rangle > 0$ for all $g \in M^\star \setminus \{a\}$, and, after scaling, $|d_g| < 1/4$ for every $g \in M^\star$; let $\mathcal{J}'$ be obtained from d as in the proof of the claim, so that $Q \cap \Delta(\mathcal{J}, \mathcal{J}') = \emptyset$. At $\mathcal{J}'$, agent i^* values A_{j^*} at $2 + \sum_{g \in D_{j^*}} d_g$, which exceeds $2 + d_a + d_g$, her value of $A_{i^*} \cup \{g\}$, for every $g \in M^\star \setminus \{a\}$, and $1 + d_a$, her largest value of $A_{i^*} \triangle R$ for every other $R \in \binom{M}{\leq 1}$, so $A \notin \mathrm{IEF1}(\mathcal{J}')$. So the claim holds for IEF1, and, as for EF1, Lemmas 11 and 16 give $\mathrm{SR}_{\mathrm{rand}}(\mathrm{IEF1}) \geq L$, and the input $(\mathcal{J}, B)$ gives $\mathrm{VR}_{\mathrm{rand}}(\mathrm{IEF1}) \geq L$, even if value queries are allowed. □

Proof of Theorem 18. Let $N \coloneqq \{1, 2, \ldots, n\}$, $M \coloneqq \{1, 2, \ldots, m\}$, and $s \coloneqq \min\{n, m\}$. For the bounds that assume $m \geq n+1$, let $m \geq n+1$, $k \coloneqq \min\{m-n, \lceil n/2 \rceil\}$, $\ell \coloneqq \lfloor (m-k)/n \rfloor$, and $M^\star \coloneqq \{1, 2, \ldots, n\ell + k\}$; then $1 \leq k < n$, $\ell \geq 1$ as $m - k \geq n$, and $M^\star \subseteq M$ as $n\ell \leq m - k$. Let $B \in \Pi_n(M)$ be given by $B_i \coloneqq M \cap \{i + qn \mid q \in \{0, 1, \ldots\}\}$, so that $|B_i \cap M^\star|$ is $\ell + 1$ if $i \leq k$ and ℓ otherwise. Let $\mathcal{J}$ be the monotone additive instance with agents N, items M, and valuations $u_i(S) \coloneqq |S \cap M^\star|$ for $S \subseteq M$ and $i \in N$. At $\mathcal{J}$, every allocation A has $M^+_{i,j} = A_j \cap M^\star$, so a complete A is EFX_0 if and only if $|A_i \cap M^\star| \geq |A_j \cap M^\star| - 1$ for all $i, j \in N$. As $|M^\star| = n\ell + k$ with $0 < k < n$, the set $\mathrm{EFX}_0(\mathcal{J})$ thus consists of the complete allocations that give k agents $\ell + 1$ items of $M^\star$ each and the other agents ℓ each, such as B.

We claim that, for every $A \in \mathrm{EFX}_0(\mathcal{J})$ and every set Q of fewer than $L \coloneqq k(\ell+1)(n-k)$ queries, some monotone additive instance $\mathcal{J}'$ with agents N and items M satisfies $Q \cap \Delta(\mathcal{J}, \mathcal{J}') = \emptyset$ and $A \notin \mathrm{EFX}_0(\mathcal{J}')$. Then Lemma 16 with $\mathcal{P}' \coloneqq \mathcal{P} \coloneqq \mathrm{EFX}_0$ gives $\mathrm{SR}_{\mathrm{rand}}(\mathrm{EFX}_0) \geq L$ even if value queries are allowed. This proves the bound on search, as $n - k \geq \lfloor n/2 \rfloor \geq n/3$, and

$k(\ell+1) \geq 2(m-n)$ if $k = m-n$, while otherwise $k \geq n/2$ and $\ell + 1 > (m-k)/n \geq (m-n)/n$; so $L \geq n(m-n)/6$.

To prove the claim, let N_1 and N_2 be the sets of agents i with $|A_i \cap M^\star| = \ell$ and $\ell+1$, respectively, and let $S_i \coloneqq A_i \cap M^\star$ for $i \in N$. Some $i^* \in N_1$ receives fewer than $k(\ell+1)$ of the queries in Q, and the restrictions to $M^\star$ of the vectors of these queries, as defined in the proof of Theorem 10, span a subspace $H \subseteq \mathbb{R}^{M^\star}$ of dimension less than $k(\ell+1)$. The $k(\ell+1)$ vectors $z_{j,g} \coloneqq \mathbf{1}_{M^\star|S_j \setminus \{g\}} - \mathbf{1}_{M^\star|S_{i^*}}$ with $j \in N_2$ and $g \in S_j$ are linearly independent: if $\sum_{j,g} \lambda_{j,g} z_{j,g} = \mathbf{0}$, then the coordinate of each $g' \in S_j$ gives $\lambda_{j,g'} = \Lambda_j \coloneqq \sum_{g \in S_j} \lambda_{j,g}$, so $\Lambda_j = (\ell+1)\Lambda_j = 0$. Hence some z_{j^*,g^*} lies outside H, and some $d \in \mathbb{R}^{M^\star}$ orthogonal to H satisfies $\langle d, z_{j^*,g^*} \rangle > 0$ and, after scaling, $|d_g| < 1$ for every $g \in M^\star$. Let $\mathcal{J}'$ be obtained from $\mathcal{J}$ by replacing u_{i^*} with the additive function that values each $g \in M^\star$ at $1 + d_g > 0$ and every other item at 0, which is monotone. The two valuations of i^* differ by d extended by zeros, which is orthogonal to the vectors of all queries in Q to i^*, so $Q \cap \Delta(\mathcal{J}, \mathcal{J}') = \emptyset$. But at $\mathcal{J}'$, agent i^* values $A_{j^*} \setminus \{g^*\}$ above A_{i^*} by $\langle d, z_{j^*,g^*} \rangle > 0$, as both contain ℓ items of $M^\star$, and $g^* \in M^+_{i^*,j^*}$ as it has a positive value, so $A \notin \mathrm{EFX}_0(\mathcal{J}')$.

For the verification bounds of order nm, let first $m \leq n$, and let $\mathcal{F}$ and A° be the family of instances and the allocation in the proof of Theorem 17, so that the agents $j \leq \lfloor m/2 \rfloor$ hold sets A°_j of $b_j \in \{2,3\}$ items that partition M and the other $n - \lfloor m/2 \rfloor \geq n/2$ agents hold nothing. As shown there, A° is EFX_0 at every member of $\mathcal{F}$, since each agent $i > \lfloor m/2 \rfloor$ has $M^+_{i,j} = \{\tau_{i,j}\}$ and values $A^\circ_j \setminus \{\tau_{i,j}\}$ at $0 = v_i(A^\circ_i)$. It is not EFX_0 at the midpoint of two distinct members, where some such i values two items of some A°_j at $1/2$ each, both of which lie in $M^+_{i,j}$, and removing either leaves the value $1/2 > 0$. Hence, if only comparison queries are allowed, Lemma 15 gives $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EFX}_0) \geq \log_2 |\mathcal{F}| = (n - \lfloor m/2 \rfloor) \sum_j \log_2 b_j \geq nm/4$, as $\log_2 b \geq b/2$ for $b \in \{2,3\}$ and $\sum_j b_j = m$.

Now let $m \geq 3$, which holds if $m > n$, let $h \coloneqq \min\{n-1, \lfloor m/2 \rfloor\}$ and $r \coloneqq m - h \geq \lceil m/2 \rceil \geq 2$, and let $G \coloneqq \{h+1, \ldots, m\}$. Let D be the complete allocation with $D_1 \coloneqq G$, $D_i \coloneqq \{i-1\}$ for $i \in \{2, \ldots, h+1\}$, and $D_i \coloneqq \emptyset$ for the other agents i. Let $\mathcal{K}$ be the monotone additive instance in which each agent $i \in \{2, \ldots, h+1\}$ has the valuation $w_i \coloneqq (r-1)\mathbf{1}_{M|\{i-1\}} + \mathbf{1}_{M|G}$ and every other agent has the valuation $\mathbf{0}$. Each such agent i values D_i, and $D_1 \setminus \{g\}$ for every $g \in G$, at $r-1$ and every other bundle at 0, so $D \in \mathrm{EFX}^+_0(\mathcal{K}) \subseteq \mathrm{EFX}_0(\mathcal{K})$. We claim that, for every set Q of fewer than hr queries, some monotone additive instance $\mathcal{K}'$ with agents N and items M satisfies $Q \cap \Delta(\mathcal{K}, \mathcal{K}') = \emptyset$ and $D \notin \mathrm{EFX}_0(\mathcal{K}') \supseteq \mathrm{EFX}^+_0(\mathcal{K}')$. As in the proof of Theorem 10, with $(\mathcal{K}, D)$ in place of $(\mathcal{J}, B)$, this gives $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EFX}_0) \geq hr$ and $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EFX}^+_0) \geq hr$ even if value queries are allowed. Indeed, some $i \in \{2, \ldots, h+1\}$ receives fewer than r of the queries in Q, and the restrictions of the vectors of these queries to $M_i \coloneqq G \cup \{i-1\}$ span a subspace $H \subseteq \mathbb{R}^{M_i}$ of dimension less than r. As for the vectors $z_{j,g}$ above, with the single block G of $r \geq 2$ items, the r vectors $z_g \coloneqq \mathbf{1}_{M_i|G\setminus\{g\}} - \mathbf{1}_{M_i|\{i-1\}}$ with $g \in G$ are linearly independent, so some z_{g^*} lies outside H. So some $d \in \mathbb{R}^{M_i}$ orthogonal to H satisfies $\langle d, z_{g^*} \rangle > 0$ and, after scaling, $|d_g| < 1$ for every $g \in M_i$; let $\mathcal{K}'$ be obtained from $\mathcal{K}$ by adding d, extended by zeros, to w_i. Then $w_i + d$ is positive on M_i and 0 elsewhere, and $Q \cap \Delta(\mathcal{K}, \mathcal{K}') = \emptyset$ as before. But at $\mathcal{K}'$, agent i values $D_1 \setminus \{g^*\}$ above D_i by $\langle d, z_{g^*} \rangle > 0$, as w_i values both at $r-1$, and $g^* \in M^+_{i,1}$ as it has a positive value, so $D \notin \mathrm{EFX}_0(\mathcal{K}')$. If $m > n$, then $h \geq n/2$, as $n - 1 \geq n/2$ and $\lfloor m/2 \rfloor \geq \lceil n/2 \rceil$, and $r \geq m/2$, so $hr \geq nm/4$. With the bound for $m \leq n$ and the upper bound $\mathrm{VR}_{\mathrm{det}}(\mathrm{EFX}_0) = O(nm)$ of Proposition 2, this proves these bounds.

For EFX^+_0, note that $h \geq s/3$, as $n - 1 \geq n/2$ and $\lfloor m/2 \rfloor \geq m/3$, so $hr \geq ms/6$ if $m \geq 3$. Moreover, let $\mathcal{I}'_0$ be the instance with agents N, items M, and all valuations $\mathbf{0}$, at which $A' \coloneqq (M, \emptyset, \ldots, \emptyset)$ is EFX^+_0, and for each $j \in N \setminus \{1\}$, let $\mathcal{I}'_j$ be obtained from $\mathcal{I}'_0$ by giving agent j the valuation $\mathbf{1}_{M|\{1\}}$. At $\mathcal{I}'_j$, the item 2 lies in $M^+_{j,1}$, as equality is allowed there, and agent j values

$M \setminus \{2\}$ at $1 > 0 = v_j\left(A'_j\right)$, so $A' \notin \mathrm{EFX}_0^+\left(\mathcal{I}'_j\right)$. The sets $\Delta\left(\mathcal{I}'_0, \mathcal{I}'_j\right)$ consist of queries to agent j, so they are pairwise disjoint, and they stay the same when A' is attached to the instances. By Lemma 12 with $\varepsilon := 0$ and unit weights, every zero-error verifier makes at least $n-1 \geq n/2$ queries in expectation on $(\mathcal{I}'_0, A')$, also with value queries. Using $hr \geq ms/6$ if $m \geq 3$ and $ms = 4 \leq 2n$ if $m = 2$, we obtain $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EFX}_0^+) \geq (ms+n)/12$, and Proposition 2 gives the matching upper bound $O(ms+n)$, which proves the bound for EFX_0^+.

For the bounded-error bound, let $m \geq n+1$ and let only comparison queries be allowed, and let $\varepsilon \in [0, 1/2)$ and $\delta := 2^{-1-|M^\star|}$. Let $\mathcal{I}_0$ be the instance with agents N, items M, and the additive valuations $v_i := \mathbf{1}_{M|M^\star}$ for $i \leq k$ and $v_i := \mathbf{1}_{M|M^\star} - \delta \sum_{g \in M^\star \setminus B_i} 2^{g-1} \mathbf{1}_{M|\{g\}}$ for $i > k$, which are monotone, as $\delta 2^{g-1} < 1/2$ for $g \in M^\star$. Then $B \in \mathrm{EFX}_0(\mathcal{I}_0)$, as $M^+_{i,j} \subseteq B_j \cap M^\star$: for $j \neq i$ and $g \in B_j \cap M^\star$, the bundle $B_j \setminus \{g\}$ has at most ℓ items of $M^\star$, so agent $i \leq k$ values it at most at $\ell < \ell + 1 = v_i(B_i)$, and agent $i > k$, who values every item of $M^\star \setminus B_i$ below 1, values it below $\ell = v_i(B_i)$. For $j \leq k$, let T_j consist of the two largest items of $B_j \cap M^\star$, and for $i > k$ and $g \in T_j$, let $\mathcal{I}_{i,g}$ be obtained from $\mathcal{I}_0$ by flipping the sign of the term of g in v_i. With g' the other item of T_j, the bundle $B_j \setminus \{g'\}$ consists of g, $\ell - 1$ smaller items of $M^\star$, and items outside $M^\star$, so, as $\sum_{g'' < g} 2^{g''-1} < 2^{g-1}$, agent i values it above $\ell = v_i(B_i)$ at $\mathcal{I}_{i,g}$, while $g' \in M^+_{i,j}$. So $B \notin \mathrm{EFX}_0(\mathcal{I}_{i,g})$ for each of the $2k(n-k)$ pairs (i,g), and no output valid for $(\mathcal{I}_0, B)$ is valid for any $(\mathcal{I}_{i,g}, B)$. Since $\delta < 2^{-|M^\star|}$, Lemma 14 applies to v_i with $c := \mathbf{1}_{M|M^\star}$, $U := M^\star \setminus B_i$, $\sigma_g := -1$ for $g \in U$, and the ranks $g \mapsto g - 1$, so $\mathcal{I}_{i,g}$ answers a comparison differently from $\mathcal{I}_0$ only if it is a comparison to agent i whose normal h satisfies $t(h) = g$. So the sets $\Delta((\mathcal{I}_0, B), (\mathcal{I}_{i,g}, B))$ are pairwise disjoint, as distinct pairs differ in the agent or in g, and by Lemma 12 with unit weights, every algorithm that verifies correctly with probability at least $1 - \varepsilon$ makes at least $(1-2\varepsilon)\,2k(n-k)$ queries in expectation on $(\mathcal{I}_0, B)$.

Finally, let $\mathcal{C}$ be the set of all antitone additive instances with agents N and items M, and allow value queries. First let $m > n$ and $r := \min\{n, m-n\}$, and let $A \in \Pi_n(M)$ give $s_i := |A_i| \geq 2$ items to each agent i of a set R of r agents, which we call *active*, and one item to each other agent, which is possible as $m - n \geq r$. Let $\mathcal{J}$ be the instance in which each active agent i values every item of A_i at -1 and every item of A_j, for $j \neq i$, at $-(s_i - 1)/s_j$, so that $v_i(A_i \setminus \{g\}) = -(s_i - 1) = v_i(A_j)$ for all $j \neq i$ and $g \in A_i$, and every other agent values every item at -1. Then $A \in \mathrm{EFX}_0(\mathcal{J})$, as $M^+_{i,j} = \emptyset$ for all $i, j \in N$, $M^-_i = A_i$ for every active i, and every other agent i has $v_i(A_i \setminus \{g\}) = 0 \geq v_i(A_j)$. For $i \in R$, let W_i be the span of the vectors $\mathbf{1}_{M|A_i \setminus \{g\}} - \mathbf{1}_{M|A_j}$ with $g \in A_i$ and $j \in N \setminus \{i\}$. Fixing $g_0 \in A_i$ and $j_0 \in N \setminus \{i\}$, it contains the $s_i - 1$ vectors $\mathbf{1}_{M|\{g\}} - \mathbf{1}_{M|\{g_0\}}$ with $g \in A_i \setminus \{g_0\}$, the $n-2$ vectors $\mathbf{1}_{M|A_j} - \mathbf{1}_{M|A_{j_0}}$ with $j \in N \setminus \{i, j_0\}$, and $\mathbf{1}_{M|A_i \setminus \{g_0\}} - \mathbf{1}_{M|A_{j_0}}$, which are linearly independent, as their restrictions to the sets A_j with $j \notin \{i, j_0\}$, to A_{j_0}, and to A_i show in turn, since all bundles are nonempty; so $\dim W_i \geq s_i + n - 2$. Let Q be a set of fewer than $\sum_{i \in R}(s_i + n - 2) = m - n + r(n-1)$ queries. Then some active agent i receives fewer than $s_i + n - 2$ of them, so the span of the vectors of these queries does not contain W_i, and some $d \in \mathbb{R}^M$ orthogonal to all of them satisfies $\left\langle d, \mathbf{1}_{M|A_i \setminus \{g\}} - \mathbf{1}_{M|A_j} \right\rangle < 0$ for some $g \in A_i$ and $j \neq i$ and, after scaling, $|d_h| < 1/m$ for every $h \in M$. Let $\mathcal{J}'$ be obtained from $\mathcal{J}$ by replacing v_i with the additive function $v_i + d$, which is negative on every item, as every item value of v_i is at most $-1/m$; then $Q \cap \Delta(\mathcal{J}, \mathcal{J}') = \emptyset$, as in the proof of Theorem 10, and $A \notin \mathrm{EFX}_0(\mathcal{J}')$, as $g \in M^-_i$ and agent i values $A_i \setminus \{g\}$ below A_j at $\mathcal{J}'$. The argument for verification in the proof of Theorem 10, with the input $(\mathcal{J}, A)$, gives $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EFX}_0) \geq m - n + r(n-1)$, even if value queries are allowed; for $m \geq 2n$, this is $m + n(n-2)$.

Next let $m \leq n$ and $k := \lfloor m/2 \rfloor \geq 1$, and let $A \in \Pi_n(M)$ give two items g_i and g'_i to each agent $i \leq k$, the remaining item, if m is odd, also to agent 1, and nothing to the other $n - k \geq 1$ agents. Let $\mathcal{J}_0$ be the instance in which each agent $i \leq k$ values g_i at -1 and every other item

at 0, and every other agent values every item at 0, and, for $i \leq k$, let $\mathcal{J}_i$ be obtained from $\mathcal{J}_0$ by changing the value of g'_i to agent i to -1. By the proof of Proposition 2, as some bundle is empty, $A \in \mathrm{EFX}_0(\mathcal{J}_0)$, since every agent has at most one item of negative value, whereas $A \notin \mathrm{EFX}_0(\mathcal{J}_i)$, since agent i has two. As $\mathcal{J}_i$ differs from $\mathcal{J}_0$ only in the valuation of agent i, the k sets $\Delta((\mathcal{J}_0, A), (\mathcal{J}_i, A))$ are pairwise disjoint, also with value queries, and Lemma 12 with unit weights gives $\mathrm{VR}_\varepsilon(\mathrm{EFX}_0) \geq (1-2\varepsilon)\lfloor m/2 \rfloor$ for every $\varepsilon \in [0, 1/2)$, and $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EFX}_0) \geq \lfloor m/2 \rfloor$ with $\varepsilon = 0$.

Proposition 2 gives the upper bound $2m + (n-2)\min\{n, (m-n)_+\}$, and the lower bounds are of the same order: $\lfloor m/2 \rfloor \geq m/3$ for $m \leq n$, $m - n + r(n-1) = nr \geq (m+nr)/3$ for $n < m < 2n$, and $m - n + r(n-1) \geq (m+n^2)/2$ for $m \geq 2n$. □

Proof of Theorem 19. Let $N \coloneqq \{1, 2, \ldots, n\}$ and $M \coloneqq \{1, 2, \ldots, m\}$. Call a pair of integers (k, ℓ) *admissible* if $1 \leq k < n$, $\ell \geq 1$, and $n\ell + k \leq m$. For $\ell \in \{1, \ldots, L\}$, the largest k such that (k, ℓ) is admissible is $n-1$ if $\ell < L$, as then $m - n\ell \geq n + q$, and q if $\ell = L$ and $q \geq 1$, and no pair with $\ell > L$ is admissible; as $q \geq 1$ or $L \geq 2$ by $m > n$, the largest product $k\ell$ of an admissible pair is thus $\max\{qL, (n-1)(L-1)\} = R - (n-1)$. Fix an admissible pair (k, ℓ) with $k\ell + n - 1 = R$, and, as in the proof of Theorem 18, let $M^\star \coloneqq \{1, 2, \ldots, n\ell + k\}$, let $B \in \Pi_n(M)$ be given by $B_i \coloneqq M \cap \{i + qn \mid q \in \{0, 1, \ldots\}\}$, and let $\mathcal{J}$ be the monotone additive instance in which every agent values each $S \subseteq M$ at $|S \cap M^\star|$. Note that $\mathcal{J}$ is homogeneous and that $\mathrm{EFX}_0(\mathcal{J})$ consists of the allocations described there. We claim that, for every $A \in \mathrm{EFX}_0(\mathcal{J})$ and every set Q of fewer than $R = k\ell + n - 1$ queries, some homogeneous monotone additive instance $\mathcal{J}'$ with agents N and items M satisfies $Q \cap \Delta(\mathcal{J}, \mathcal{J}') = \emptyset$ and $A \notin \mathrm{EFX}_0(\mathcal{J}')$. As in the proof of Theorem 10, this gives $\mathrm{SR}_{\mathrm{rand}}(\mathrm{EFX}_0) \geq R$ even if value queries are allowed, and $R \geq \lceil m/2 \rceil$, since R is an integer with $2R \geq m$. Indeed, if $m < 2n$, then $L = 1$, $R = n - 1 + q = m - 1$, and $m \geq 3$; otherwise, the pair $(n-1, \lfloor (m+1)/n \rfloor - 1)$ is admissible, and $\lfloor (m+1)/n \rfloor \geq (m-n+2)/n$, so $R \geq (n-1)\lfloor (m+1)/n \rfloor$ and $2R - m \geq (n-2)(m-2n+2)/n \geq 0$.

To prove the claim, let N_1, N_2, and S_i be as in the proof of Theorem 18, write $d(S) \coloneqq \sum_{g \in S} d_g$ for $d \in \mathbb{R}^{M^\star}$ and $S \subseteq M^\star$, and let Z be the set of vectors $\mathbf{1}_{M^\star | S_j \setminus \{g\}} - \mathbf{1}_{M^\star | S_i}$ with $i \in N_1$, $j \in N_2$, and $g \in S_j$. Every $d \in \mathbb{R}^{M^\star}$ orthogonal to Z satisfies $d_g = d(S_j) - d(S_i)$ for all such i, j, and g, so it is constant on each S_j, say c_j, and then $d(S_i) = \ell c_j$, which makes all c_j equal to one value c, as $N_1 \neq \emptyset$ and $\ell \geq 1$. Such a d is thus determined by c and its restrictions to the sets S_i with $i \in N_1$, whose sums are ℓc, so these vectors d form a space of dimension at most $1 + (n-k)(\ell - 1)$, and Z spans a space W of dimension at least $|M^\star| - 1 - (n-k)(\ell-1) = R$. The restrictions to $M^\star$ of the vectors of all queries in Q span a space of dimension less than R, which thus does not contain W, so some $z = \mathbf{1}_{M^\star | S_j \setminus \{g\}} - \mathbf{1}_{M^\star | S_i} \in Z$ lies outside it, and some d orthogonal to it satisfies $\langle d, z \rangle > 0$ and, after scaling, $|d_{g'}| < 1$ for every $g' \in M^\star$. Let $\mathcal{J}'$ be the homogeneous monotone additive instance whose common valuation values each $g' \in M^\star$ at $1 + d_{g'} > 0$ and every other item at 0; as every query in Q concerns this valuation, $Q \cap \Delta(\mathcal{J}, \mathcal{J}') = \emptyset$, as in the proof of Theorem 18. But at $\mathcal{J}'$, agent i values $A_j \setminus \{g\}$ above A_i by $\langle d, z \rangle > 0$, as both contain ℓ items of $M^\star$, and $g \in M^+_{i,j}$ as it has a positive value, so $A \notin \mathrm{EFX}_0(\mathcal{J}')$.

For verification, let $r \coloneqq m - n + 1 \geq 2$, let $H \coloneqq \{1, 2, \ldots, n-1\}$ and $G \coloneqq M \setminus H$, so that $|G| = r$, and let $\mathcal{J}_0$ be the homogeneous monotone additive instance whose common valuation values each $a \in H$ at $r - 1$ and each $g \in G$ at 1. The allocation A^0 that gives each $a \in H$ to agent a and G to agent n is in $\mathrm{EFX}_0(\mathcal{J}_0)$, as $r - 1 = |G \setminus \{g\}|$ for every $g \in G$ and removing the item of a singleton bundle leaves the value 0. The vectors $z_{a,g} \coloneqq \mathbf{1}_{M | G \setminus \{g\}} - \mathbf{1}_{M | \{a\}}$ with $a \in H$ and $g \in G$ span a space of dimension $m - 1$: for a fixed a, the r vectors $z_{a,g}$ restricted to G form the all-ones matrix minus the identity matrix, which is invertible as $r \geq 2$, and the differences $z_{a,g} - z_{a',g} = \mathbf{1}_{M | \{a'\}} - \mathbf{1}_{M | \{a\}}$ add the $n - 2$ dimensions of the vectors supported on H with zero sum. So, for every set Q of

fewer than $m-1$ queries, some $z_{a,g}$ lies outside the span of the vectors of the queries in Q, and some $d \in \mathbb{R}^M$ orthogonal to that span satisfies $\langle d, z_{a,g}\rangle > 0$ and, after scaling, $\left|d_{g'}\right| < 1$ for every $g' \in M$. The homogeneous monotone additive instance $\mathcal{J}_0'$ whose common valuation adds d to that of $\mathcal{J}_0$ satisfies $Q \cap \Delta(\mathcal{J}_0, \mathcal{J}_0') = \emptyset$, but at $\mathcal{J}_0'$, agent a values $A_n^0 \setminus \{g\}$ above A_a^0 by $\langle d, z_{a,g}\rangle > 0$, and $g \in M_{a,n}^+$, so $A^0 \notin \mathrm{EFX}_0(\mathcal{J}_0')$. As in the proof of Theorem 10, this gives $\mathrm{VR}_{\mathrm{rand}}(\mathrm{EFX}_0) \geq m-1$ even if value queries are allowed.

Lastly, let v be the common valuation of the unknown instance and A the given allocation, which is rejected without queries if it is not complete. As v is monotone and additive, $M_i^- = \emptyset$, and $A \in \mathrm{EFX}_0$ if and only if $v(A_{i^*}) \geq v(A_j \setminus \{g\})$ for all $j \in N$ and $g \in A_j$ with $v(\{g\}) > v(\emptyset)$, where A_{i^*} is a bundle of minimum value. An empty bundle is one, and otherwise $n-1$ queries find one; then two queries for each $g \in M$ test whether $v(\{g\}) > v(\emptyset)$ and the inequality for the bundle $A_j \ni g$, so $\mathrm{VR}_{\mathrm{det}}(\mathrm{EFX}_0) \leq 2m+n-1$. □

B.3 Section 3.3

Proof of Theorem 20. By Definition 9, $v_i(A_i) < \mu_{v_i}(d)$ if and only if some $A' \in \Pi_d(M)$ has all its parts in $\mathcal{G}_i = \{T \subseteq M \mid v_i(T) > v_i(A_i)\}$, which Algorithm 1 tests for each $i \in N'$ after computing $\mathcal{G}_i$ with $O\left(|M|^2 \log |M|\right)$ queries by Theorem 2. In the homogeneous case, all agents share the valuation v_1 and hence the share $\mu_{v_1}(d)$, so only an agent with a bundle of least value needs to be checked; at most $\min\{|N|, |M|+1\}$ of the bundles A_i are distinct, as at most $|M|$ of them are nonempty, so at most $|M|$ comparisons find a bundle of minimum value. □

Shares. Let M be a finite set with $|M| \geq 2$ and $v : 2^M \to \mathbb{R}$ be additive, so that $v(\emptyset) = 0$; then v is monotone if and only if $v(\{g\}) \geq 0$ for every $g \in M$, in which case $0 \leq v(S) \leq v(T)$ for all $S \subseteq T \subseteq M$. For $X \subseteq M$ and $j \in \{1, 2, \ldots\}$, let $\mu_v(j; X) \coloneqq \max_{B \in \Pi_j(X)} \min_{t \in \{1,2,\ldots,j\}} v(B_t)$, which is attained as $\Pi_j(X)$ is finite and nonempty, so that $\mu_v(j) = \mu_v(j; M)$ in Definition 9. For $k \in \{1, 2, \ldots\}$, $\rho_v(k)$ is the residual maximin share of Definition 10 for $d \coloneqq k$ parts. A *configuration* for k is a triple (R, j, B) with $R \subseteq M$, $j \in \{1, 2, \ldots, k\}$, and $B \in \Pi_{k-j}(R)$, and $\tau \in \mathbb{R}$ is *feasible* for k if, for every configuration (R, j, B) for k, $\max_{t \in \{1,2,\ldots,k-j\}} v(B_t) < \tau$ implies $\mu_v(j; M \setminus R) \geq \tau$, where $\max \emptyset = -\infty$. By the footnote to Definition 10, with j and k in place of its k and d, the values feasible for k are those at most $\rho_v(k)$, and

$$\rho_v(k) = \min_{(R,j,B)} \max \left\{ \max_{t \in \{1,2,\ldots,k-j\}} v(B_t), \mu_v(j; M \setminus R) \right\}, \tag{27}$$

where the minimum is over all configurations for k. As $\Pi_0(R) = \emptyset$ for $R \neq \emptyset$ and $\Pi_0(\emptyset) = \{()\}$, the only configuration with $j = k$ is $(\emptyset, k, ())$, whose term in Eq. (27) is $\mu_v(k)$. For $K \subseteq M$ with $|K| \geq 2$, all these notions apply to the restriction of v to 2^K as well.

Lemma 21. *Let M be a finite set with $|M| \geq 2$ and $v : 2^M \to \mathbb{R}$ be additive. Then $\rho_v(k) \leq \mu_v(k)$ for every $k \in \{1, 2, \ldots\}$, and $\rho_v(2) = \mu_v(2)$. Moreover, $\mu_v(d) = \min\{0, \mu_v(|M|)\}$ for every integer $d > |M|$. If v is monotone, then $\rho_v(k) \geq 0$ for every $k \in \{1, 2, \ldots\}$, $\mu_v(|M|) = \rho_v(|M|) = \min_{g \in M} v(\{g\})$, and $\mu_v(d) = \rho_v(d) = 0$ for every integer $d > |M|$.*

Proof. The configuration $(\emptyset, k, ())$ gives $\rho_v(k) \leq \mu_v(k)$ by Eq. (27). For $k = 2$, every other configuration is $(R, 1, (R))$ for some $R \subseteq M$, as $\Pi_1(X) = \{(X)\}$ for every $X \subseteq M$, and its term in Eq. (27) is $\max\{v(R), v(M \setminus R)\} \geq v(M)/2$, as $v(R) + v(M \setminus R) = v(M)$. Since $\min\{v(B_1), v(B_2)\} \leq v(M)/2$ for every $B \in \Pi_2(M)$, we have $v(M)/2 \geq \mu_v(2)$, so $\rho_v(2) = \mu_v(2)$.

Let $d > |M|$. Every $B \in \Pi_d(M)$ has an empty part, worth 0, and at most $|M|$ nonempty parts, which, completed by empty parts, form some $B' \in \Pi_{|M|}(M)$ each of whose parts is a part

of B; hence $\min_t v(B_t) \le \min\{0, \min_t v(B'_t)\} \le \min\{0, \mu_v(|M|)\}$. Conversely, completing some $B' \in \Pi_{|M|}(M)$ with $\min_t v(B'_t) = \mu_v(|M|)$ by $d - |M|$ empty parts gives a d-partition whose least part value is $\min\{0, \mu_v(|M|)\}$.

Let v be monotone and $\lambda_0 \coloneqq \min_{g \in M} v(\{g\}) \ge 0$. The value 0 is feasible for every k, as a configuration with $j < k$ has a part, worth at least 0, and $\mu_v(k) \ge 0$; hence $\rho_v(k) \ge 0$. An $|M|$-partition of M has an empty part, worth $0 \le \lambda_0$, or consists of the singletons, so $\mu_v(|M|) = \lambda_0$. If $\lambda_0 > 0$, then λ_0 is feasible for $|M|$: in a configuration (R, j, B) whose parts B_t are all worth less than λ_0, these parts are empty, as every nonempty bundle is worth at least λ_0, so $R = \emptyset$, and $\mu_v(j; M) \ge \lambda_0$, as M can be split into $j \le |M|$ nonempty parts. Together with $0 \le \rho_v(|M|) \le \mu_v(|M|)$, this gives $\rho_v(|M|) = \lambda_0$. For $d > |M|$, the second claim gives $\mu_v(d) = \min\{0, \lambda_0\} = 0$, and then $0 \le \rho_v(d) \le \mu_v(d)$ gives $\rho_v(d) = 0$. □

For nonnegative values and $d > |M|/2$, the following lemma reduces both shares to the $2(|M| - d)$ items of least value.

Lemma 22. *Let M be a finite set, $v : 2^M \to \mathbb{R}$ be monotone additive, and d be an integer with $|M|/2 < d < |M|$. Let $s \coloneqq |M| - d$, let $K \subseteq M$ be a set of $2s$ items such that $v(\{h\}) \le v(\{g\})$ for all $h \in K$ and $g \in L \coloneqq M \setminus K$, let $\lambda \coloneqq \min_{g \in L} v(\{g\})$, and let w be the restriction of v to 2^K. Then $\mu_v(d) = \min\{\lambda, \mu_w(s)\}$ and $\rho_v(d) = \min\{\lambda, \rho_w(s)\}$.*

Proof. As $1 \le s < |M|/2$, we have $|K| = 2s \ge 2$ and $\ell \coloneqq |L| = d - s \ge 1$. Every $T \subseteq M$ that meets L satisfies $v(T) \ge \lambda$; moreover, $\lambda \ge 0$ and $\mu_w(s) \ge 0$ as v is monotone, and $\rho_w(s) \ge 0$ by Lemma 21.

For the first identity, an s-partition of K attaining $\mu_w(s)$, together with the ℓ singletons $\{g\}$ with $g \in L$, forms a d-partition of M whose least part value is $\min\{\lambda, \mu_w(s)\}$, so $\mu_v(d) \ge \min\{\lambda, \mu_w(s)\}$. Conversely, let $B \in \Pi_d(M)$ attain $\mu \coloneqq \mu_v(d)$, and suppose that $\mu > 0$, as otherwise there is nothing to prove. Then all parts of B are nonempty, and since $\sum_t (|B_t| - 1) = |M| - d = s$, the parts with at least two items contain at most $2s$ items in all. Hence, if a items of L lie in such parts, then at most $2s - a$ items of K do, so at least a items of K form parts by themselves. Exchanging each of these a items of L with a distinct one of these items of K gives a d-partition B' of M whose parts are all still worth at least μ: each new part $\{g\}$ with $g \in L$ replaces a part $\{h\}$ with $v(\{g\}) \ge v(\{h\}) \ge \mu$, and every other changed part contains such an item h. In B', every item of L forms a part by itself, and the other $d - \ell = s$ parts form an s-partition of K, so $\mu \le \lambda$ and $\mu \le \mu_w(s)$.

For the second identity, $\rho_v(d) \le \mu_v(d) \le \lambda$ by Lemma 21 and the first identity. Let $0 < \tau \le \lambda$; we show that τ is feasible for d with respect to v if and only if it is feasible for s with respect to w. Let (R, j, B) be a configuration for d with respect to v whose parts B_t are all worth less than τ; only such configurations constrain τ. These parts avoid L, as $\tau \le \lambda$, so $R \subseteq K$ and $M \setminus R = L \cup (K \setminus R)$. If $j \le \ell$, then $\mu_v(j; M \setminus R) \ge \lambda \ge \tau$, as we may split L into j nonempty parts and add $K \setminus R$ to one of them. If $j > \ell$, then $\mu_v(j; M \setminus R) \ge \tau$ if and only if $\mu_w(j - \ell; K \setminus R) \ge \tau$. Indeed, adding the ℓ singletons $\{g\}$ with $g \in L$ to a $(j - \ell)$-partition of $K \setminus R$ whose parts are all worth at least τ gives such a j-partition of $M \setminus R$; conversely, in a j-partition of $M \setminus R$ whose parts are all worth at least τ, at most ℓ parts meet L, so at least $j - \ell$ parts are subsets of $K \setminus R$, and adding the remaining items of $K \setminus R$ to one of $j - \ell$ such parts gives a $(j - \ell)$-partition of $K \setminus R$ whose parts are all worth at least τ, by monotonicity. Since $d - j = s - (j - \ell)$, the map $(R, j, B) \mapsto (R, j - \ell, B)$ is a bijection from the configurations for d with respect to v with $R \subseteq K$ and $j > \ell$ onto the configurations for s with respect to w, and it keeps the parts B_t. So a configuration for d that constrains τ either is satisfied automatically or is mapped to a configuration for s that constrains τ, every configuration for s that constrains τ arises in this way, and corresponding configurations

impose the same condition on τ. Hence the values in $(0, \lambda]$ feasible for d form both the interval $(0, \rho_v(d)]$, as $\rho_v(d) \leq \lambda$, and the interval $(0, \min\{\lambda, \rho_w(s)\}]$; as both endpoints are nonnegative, they are equal. □

Lemma 23. *Let M be a finite set with $|M| \geq 2$, let $d \in \{1, 2, \ldots\}$, and let σ be either μ or ρ. Given a queryable additive function $v : 2^M \to \mathbb{R}$, one can find $S^* \subseteq M$ with $v(S^*) = \sigma_v(d)$ and compute $(\operatorname{sign}(v(T) - \sigma_v(d)))_{T \subseteq M}$ using $O\left(|M|^3 \log |M|\right)$ comparison queries. If v is monotone, then one can compute $\mathcal{F} \coloneqq \{T \subseteq M \mid v(T) \geq \sigma_v(d)\}$ using $O\left(|M| + (|M| - d)^3 \log(|M| - d + 2)\right)$ comparison queries if $|M|/2 < d \leq |M|$, and without queries if $d > |M|$.*

Proof. For the first claim, we check the hypothesis of Theorem 6 for $\tau(v) \coloneqq \sigma_v(d)$. Each term in Eq. (27) is a value $v(T)$ with $T \subseteq M$, and so is $\mu_v(d)$, so $\tau(v) \in \{v(T) \mid T \subseteq M\}$. For $S \subseteq M$, $\mu_v(d) \geq v(S)$ if and only if some $B \in \Pi_d(M)$ has $v(B_t) \geq v(S)$ for every t, and, by Eq. (27), $\rho_v(d) \geq v(S)$ if and only if every configuration (R, j, B) for d has a part B_t with $v(B_t) \geq v(S)$ or admits some $B' \in \Pi_j(M \setminus R)$ with $v(B'_t) \geq v(S)$ for every t. The same holds with strict inequalities, and all these conditions depend only on $(\operatorname{sign}(v(T) - v(S)))_{T \subseteq M}$, so Theorem 6 gives the first claim.

Let v be monotone. If $d > |M|$, then $\sigma_v(d) = 0$ by Lemma 21, so $\mathcal{F} = 2^M$. If $d = |M|$, then $\sigma_v(d) = \min_{g \in M} v(\{g\})$ by Lemma 21; $|M| - 1$ queries find $g_0 \in M$ attaining this minimum, and one more query tells whether $v(\{g_0\}) \leq v(\emptyset)$, in which case $\sigma_v(d) = 0$ and $\mathcal{F} = 2^M$, and otherwise $\mathcal{F} = 2^M \setminus \{\emptyset\}$, as every nonempty $T \subseteq M$ is worth at least $v(\{g_0\})$.

Let $|M|/2 < d < |M|$ and $s \coloneqq |M| - d$. Ordering the items by their values $v(\{g\})$, with ties broken by a fixed order of M, linear-time selection [32] finds a set K of $2s$ items as in Lemma 22 with $O(|M|)$ comparisons of items, each of which costs at most two queries, and $|M| - 2s - 1$ more queries find $g_\lambda \in L \coloneqq M \setminus K$ with $v(\{g_\lambda\}) = \lambda \coloneqq \min_{g \in L} v(\{g\})$. The restriction w of v to 2^K is a queryable additive function, and $|K| = 2s \geq 2$, so the first claim, applied to w, finds $S^*_K \subseteq K$ with $v(S^*_K) = \sigma_w(s)$ and $(\operatorname{sign}(v(T) - \sigma_w(s)))_{T \subseteq K}$ with $O\left(s^3 \log(s + 2)\right)$ queries. By Lemma 22, $\sigma_v(d) = \min\{\lambda, \sigma_w(s)\}$, and one query tells whether $v(S^*_K) \leq v(\{g_\lambda\})$, that is, whether $\sigma_w(s) \leq \lambda$. Every $T \subseteq M$ that meets L belongs to $\mathcal{F}$, since $v(T) \geq \lambda \geq \sigma_v(d)$. If $\sigma_w(s) \leq \lambda$, then $\sigma_v(d) = \sigma_w(s)$, and the sign vector above tells which $T \subseteq K$ belong to $\mathcal{F}$. Otherwise, $\sigma_v(d) = v(\{g_\lambda\})$, and Theorem 2, applied to the restriction of v to $2^{K \cup \{g_\lambda\}}$ with the pivot $\{g_\lambda\}$, tells which $T \subseteq K$ belong to $\mathcal{F}$ with $O\left(s^2 \log(s + 2)\right)$ queries. The total is $O\left(|M| + s^3 \log(s + 2)\right)$. □

Proof of Theorem 21. Let $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$ be the unknown instance, so that $|N| \leq n$ and $|M| \leq m$, and let $\mathcal{F}_i \coloneqq \{T \subseteq M \mid v_i(T) \geq \mu_{v_i}(d)\}$ for each $i \in N$. By Definition 9, an allocation $A \in \mathcal{A}(\mathcal{I})$ belongs to 1-out-of-d-MMS$(\mathcal{I})$ if and only if A is complete and $A_i \in \mathcal{F}_i$ for every $i \in N$. The set $\mathcal{A}_0(\mathcal{I})$ is finite and depends only on N and M, so, once every $\mathcal{F}_i$ is known, one can find such an allocation, or verify that none exists, without further queries. It therefore suffices to compute every $\mathcal{F}_i$, and only $\mathcal{F}_1$ if $\mathcal{I}$ is homogeneous, as then $\mathcal{F}_i = \mathcal{F}_1$ for every $i \in N$.

For each $i \in N$, Theorem 7 computes the signs of $v_i(T) - \mu_{v_i}(d)$ for all $T \subseteq M$, and hence $\mathcal{F}_i$, with $O\left(|M|^2 \min\{d, |M|\} \log \min\{d, |M|\} \log |M|\right)$ queries, and Lemma 23 does so with $O\left(|M|^3 \log |M|\right)$ queries. As $|M| \leq m$, the resulting algorithms make $O\left(nm^2 \min\{d, m\} \log \min\{d, m\} \log m\right)$ and $O\left(nm^3 \log m\right)$ queries, and $\mathrm{SR}_{\mathrm{det}}$(1-out-of-$d$-MMS) is at most either; for $d = 2$, the former is $O\left(nm^2 \log m\right)$.

Let the instances of $\mathcal{C}$ be monotone. If $m/2 < d \leq m$, then $d > |M|/2$, so Lemma 23 computes each $\mathcal{F}_i$ without queries if $d > |M|$, and otherwise with $O\left(|M| + s^3 \log(s + 2)\right)$ queries, where $s \coloneqq |M| - d \leq m - d$, which gives the second bound. If $n(m - d + 1) \leq m$ and $d \leq |M|$, then $|N|(|M| - d + 1) \leq |M|$, as $|N| \leq n$ and $x \mapsto (n - 1)x - n(d - 1)$ is nondecreasing, so the agents

can be given disjoint bundles X_i of $|M| - d + 1$ items each, and the remaining items to any agent, without queries. For every $B \in \Pi_d(M)$, the $d-1$ items outside X_i meet at most $d-1$ parts of B, so some part B_t lies inside X_i, and $v_i(X_i) \geq v_i(B_t) \geq \min_{t'} v_i(B_{t'})$ by monotonicity; hence every agent's bundle, which contains X_i, is worth at least $\mu_{v_i}(d)$. If $d > |M|$, then $\mu_{v_i}(d) = 0$ for every $i \in N$ by Lemma 21, so every complete allocation qualifies without queries.

□

Algorithm 13 An algorithm for RMMS verification.

Input: The sets of agents N and items M of an unknown additive instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$ in a set $\mathcal{C}$ of instances and an allocation $A \in \mathcal{A}(\mathcal{I})$.

Output: Either an assertion that $v_i(A_i) \geq \rho_{v_i}(|N|)$ for every $i \in N$ or $i \in N$ such that $\rho_{v_i}(|N|) > v_i(A_i)$.

$N' \leftarrow \{i^*\}$ for some $i^* \in \arg\min_{i \in N} v_1(A_i)$, found with comparisons of v_1, if $\mathcal{C}$ consists of homogeneous instances, and $N' \leftarrow N$ otherwise.
for all $i \in N'$ **do**
    $\mathcal{G}_i \leftarrow \{T \subseteq M \mid v_i(T) > v_i(A_i)\}$ using Theorem 2.
    if $\left(\Pi_{|N|-k}(R) \cap \left(2^M \setminus \mathcal{G}_i\right)^{|N|-k} = \emptyset \ \vee \ \Pi_k(M \setminus R) \cap \mathcal{G}_i^k \neq \emptyset\right)$ for every $(R, k) \in 2^M \times \{1, 2, \ldots, |N|\}$ **then**
        return i.
return an assertion that $v_i(A_i) \geq \rho_{v_i}(|N|)$ for every $i \in N$.

Proof of Theorem 22. By Eq. (27) with $v \coloneqq v_i$ and $k \coloneqq |N|$, $\rho_{v_i}(|N|) > v_i(A_i)$ if and only if every configuration (R, j, B) for $|N|$ has a part $B_t \in \mathcal{G}_i = \{T \subseteq M \mid v_i(T) > v_i(A_i)\}$ or admits some $B' \in \Pi_j(M \setminus R)$ with all its parts in $\mathcal{G}_i$. As the latter does not depend on B, this is the condition that Algorithm 13 tests for every (R, j), with j for its k, where $\Pi_0(\emptyset) = \{()\}$ and $\Pi_0(R) = \emptyset$ for $R \neq \emptyset$, after computing $\mathcal{G}_i$ with $O\left(|M|^2 \log |M|\right)$ queries by Theorem 2. The homogeneous case is as in the proof of Theorem 20, as all agents share the residual share $\rho_{v_1}(|N|)$. □

Lemma 24. *For every instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$, with arbitrary valuation functions $v_i : 2^M \to \mathbb{R}$, $\mathrm{RMMS}(\mathcal{I}) \neq \emptyset$.*

Proof. Let $\tau_i \coloneqq \rho_{v_i}(|N|)$ for $i \in N$, which satisfies the condition in Eq. (14), as the maximum there is attained. We use the following fact: a bipartite graph between k agents and k bags in which some agent u is adjacent to every bag has a nonempty matching X such that no agent unmatched by X is adjacent to a bag matched by X. Indeed, let X_0 be a maximum matching, and let $X \coloneqq X_0$ if it matches every agent. Otherwise, let Z be the set of vertices reachable from the agents unmatched by X_0 along paths that alternate between edges outside and inside X_0. Every bag in Z is matched by X_0, as X_0 is maximum, and its partner lies in Z. If u were in Z, every bag would be in Z and hence matched, so X_0 would match all k agents; hence $u \notin Z$, and u is matched by X_0, as unmatched agents lie in Z. Let X consist of the edges of X_0 outside Z, which include the edge at u; every agent unmatched by X lies in Z, and so do all its neighbors, whereas no bag matched by X does.

We allocate in rounds and keep the invariant that every remaining agent i values each bundle allocated so far below τ_i. In a round, let N' be the set of the remaining agents, $k \coloneqq |N'| \geq 1$, and R the union of the bundles allocated so far, which form some $B \in \Pi_{|N|-k}(R)$, and pick any $i_0 \in N'$. By the invariant, every part of B is worth less than τ_{i_0} to i_0, so Eq. (14) gives $B' \in \Pi_k(M \setminus R)$ with $v_{i_0}(B'_t) \geq \tau_{i_0}$ for every t. In the bipartite graph that joins each $i \in N'$ to each bag B'_t with

$v_i(B'_t) \geq \tau_i$, agent i_0 is adjacent to every bag, so the fact gives a nonempty matching X, and each matched bag goes to its matched agent. The invariant is preserved, as the agents unmatched by X are adjacent to no matched bag, and each round removes at least one agent. In the last round, X matches all k remaining agents and hence all k bags, so every item is allocated, and every agent i receives a bundle worth at least τ_i. □

Proof of Theorem 23. Let $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$ be the unknown instance, so that $|N| \leq n$ and $|M| \leq m$. By Definition 10, an allocation $A \in \mathcal{A}(\mathcal{I})$ belongs to $\mathrm{RMMS}(\mathcal{I})$ if and only if A is complete and $A_i \in \mathcal{R}_i \coloneqq \{T \subseteq M \mid v_i(T) \geq \rho_{v_i}(|N|)\}$ for every $i \in N$, so, as in the proof of Theorem 21, it suffices to compute every $\mathcal{R}_i$, and only $\mathcal{R}_1$ if $\mathcal{I}$ is homogeneous. Lemma 23 with $\sigma \coloneqq \rho$ and $d \coloneqq |N|$ does so with $O\left(|M|^3 \log |M|\right) \subseteq O\left(m^3 \log m\right)$ queries for each agent. If $|N| = 2$, then $\rho_{v_i}(2) = \mu_{v_i}(2)$ for each $i \in N$ by Lemma 21, so $\mathrm{RMMS}(\mathcal{I}) = \text{1-out-of-2-MMS}(\mathcal{I})$, and each $\mathcal{R}_i$ is the family $\mathcal{F}_i$ of the proof of Theorem 21 for $d = 2$, which Theorem 7 computes with $O\left(m^2 \log m\right)$ queries.

By Lemma 24, such an allocation always exists, so the offline choice never reports emptiness.

Let $\mathcal{I}$ be homogeneous. Lemma 23 with $\sigma \coloneqq \mu$ and $d \coloneqq |N|$ computes the bundles $T \subseteq M$ with $v_1(T) \geq \mu_{v_1}(|N|)$ using $O\left(m^3 \log m\right)$ queries, from which some $B \in \Pi_{|N|}(M)$ with $\min_t v_1(B_t) = \mu_{v_1}(|N|)$ is found without further queries, and Theorem 7 finds such a B with $O\left(m^2 \min\{n, m\} \log \min\{n, m\} \log m\right)$ queries. Giving its parts to the agents is a complete allocation in $\mathrm{RMMS}(\mathcal{I})$, as $\mu_{v_1}(|N|) \geq \rho_{v_i}(|N|)$ for every $i \in N$ by Lemma 21.

Let $\mathcal{C}$ consist of monotone additive instances with n agents, so that $|N| = n$. If $n \geq |M|$, then $n(|M| - n + 1) \leq |M|$, so, as in the proof of Theorem 21 with $d \coloneqq n$, giving the items to distinct agents gives every agent i a bundle worth at least $\mu_{v_i}(n) \geq \rho_{v_i}(|N|)$ without queries; this covers the case $n \geq m$. If $m/2 < n < |M|$, then $n > |M|/2$, so Lemma 23 with $d \coloneqq n$ computes each $\mathcal{R}_i$ with $O\left(|M| + s^3 \log(s+2)\right)$ queries, where $s \coloneqq |M| - n \leq m - n$, and only $\mathcal{R}_1$ is needed if $\mathcal{I}$ is homogeneous, which gives the remaining bound. □

Proof of Corollary 5. Let K, E, and $\hat{u} \in \mathbb{R}^M$ for $u \in \mathbb{R}^K$ with positive entries be as in the proof of Lemma 17, so that $|K| = k$ and $|E| = n - 2$. The proof of Lemma 17 uses only the instances whose common valuation is $\hat{u}$ for some such u, and only that the algorithm decides EF correctly on them. We show that, if u is balanced, every $A \in \mathrm{RMMS}_m(\mathcal{I})$ of the instance $\mathcal{I}$ with the valuation $\hat{u}$ is complete, gives every item of E as a bundle by itself, and splits K evenly between the two other bundles. Then running a search algorithm for $\mathcal{P}$, checking that its output has this form, and comparing its two other bundles in both directions decides EF on these instances with two more comparisons and no more error, as the check also fails if u is not balanced; so that proof gives $\mathrm{SR}_\varepsilon(\mathcal{P}) \geq ck^2 - 2$, which is $\Omega\left(k^2\right)$ if k is large. For bounded k, let u give one item $g \in K$ the value $k - 1$ and the others the value 1, so that u is balanced and the only even split of K is $\{g\}$ against $K \setminus \{g\}$; two such instances with distinct g have nonempty, disjoint sets $\mathcal{P}(\mathcal{I})$, as shown below, so Lemma 12 gives $\mathrm{SR}_\varepsilon(\mathcal{P}) \geq 1 - 2\varepsilon$.

So let u be balanced, $H \coloneqq u(K)/2$, and $v \coloneqq \hat{u}$, so that $v(M) = nH$ and every item of E is worth H. The value $\tau \coloneqq H$ satisfies the condition in Eq. (14) for every agent, so $\rho_{v_i}(|N|) \geq H$. Indeed, let $R \subseteq M$, $j \in \{1, 2, \ldots, n\}$, and $B \in \Pi_{n-j}(R)$ have $v(B_t) < H$ for every t, so that no part of B meets E, and $R \subseteq K$. If $j \leq n - 2$, splitting E into j nonempty parts and adding $K \setminus R$ to one of them gives a j-partition of $M \setminus R$ whose parts are all worth at least H. If $j = n - 1$, then R is the only part of B, so $v(K \setminus R) = 2H - v(R) > H$, and $K \setminus R$ with the singletons of E gives such a partition. If $j = n$, then $R = \emptyset$, as $\Pi_0(R) = \emptyset$ for $R \neq \emptyset$, and an even split of K with the singletons of E gives one. Hence every $A \in \mathrm{RMMS}_m(\mathcal{I})$ gives every agent a bundle worth at

least H, and, as $v(M) = nH$ and all item values are positive, A is complete, every bundle is worth exactly H, a bundle with an item of E has no other item, and the two other bundles split K evenly.

The examples satisfy the assumption: 1-out-of-n-MMS $(\mathcal{I}) \subseteq$ RMMS $(\mathcal{I}) \subseteq$ RMMS$_m(\mathcal{I})$ by Lemma 21, 1-out-of-n-MMS $(\mathcal{I})$ contains every allocation whose bundles form an n-partition of M that attains $\mu_v(n)$ for the common valuation v, and EFL $\cap$ RMMS and EFX$_m^+ \cap$ RMMS$_m$ are nonempty on monotone additive instances [65], [6, Proposition 1.1]. So are EFX$_0^+ \cap$ PO $\cap$ 1-out-of-n-MMS and, if $n = 2$, EFX$_0 \cap$ PO $\cap$ 1-out-of-2-MMS on these instances, which lie in 1-out-of-n-MMS: a partition that maximizes its sorted vector of bundle values lexicographically gives an allocation that is EFX, which is EFX$_0^+$ as all item values are positive, and gives every agent her maximin share [129], and every complete allocation of a homogeneous instance is PO. □

B.4 Section 3.4

Proof of Theorem 24. Given Definition 11 and Theorem 2, we confirm that Algorithm 2 satisfies the claimed guarantees. □

Remark 13. On monotone additive instances, the agents with empty bundles need not run Theorem 2, which helps when $n > m$. Let $k \coloneqq \min\{n, m\}$. If $\mathcal{C}$ consists of monotone additive instances, then $\mathrm{VR}_{\mathrm{det}}(\mathrm{PO}) = O\left(n + km^2 \log m\right)$, by a verifier that also returns a Pareto improvement on every complete allocation that it rejects. It applies Theorem 2 only to the at most k agents with nonempty bundles, which also tells which items their owners value at 0, and asks every other agent, with one comparison, whether she values the set Z of these items at 0; if all of them do, a Pareto improvement exists only if one exists among the owners alone.

Proof of Remark 13. Let $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$ be the unknown instance and $A \in \mathcal{A}(\mathcal{I})$; if A is not complete, reject it without queries. Otherwise, let $O \coloneqq \{i \in N \mid A_i \neq \emptyset\}$, so that $|O| \leq \min\{|N|, |M|\}$, let $o(g)$ be the agent who holds g in A, and run Theorem 2 with the pivot A_i for each $i \in O$, with $O\left(|M|^2 \log |M|\right)$ comparisons each. The signs of $v_i(A_i \setminus \{g\}) - v_i(A_i)$ for $g \in A_i$ give $Z \coloneqq \left\{g \in M \mid v_{o(g)}(\{g\}) = 0\right\}$. If $Z \neq \emptyset$, compare Z with $\emptyset$ for each agent $j \in N \setminus O$; if $v_j(Z) > 0$, moving all items of Z to j is a Pareto improvement on A, as their owners value them at 0. Otherwise, search offline, using the signs given by Theorem 2, for a Pareto improvement on A that gives $\emptyset$ to every agent of $N \setminus O$; return it if one exists, and accept A otherwise. This makes $O\left(|N| + \min\{|N|, |M|\}\, |M|^2 \log |M|\right)$ comparisons.

For correctness, suppose that $v_j(Z) = 0$ for every $j \in N \setminus O$ and that A' is a Pareto improvement on A. Let A'' be obtained from A' by returning every item that A' gives to an agent of $N \setminus O$ to its owner $o(g)$ in A. Each $i \in O$ satisfies $A_i'' \supseteq A_i'$, so $v_i(A_i'') \geq v_i(A_i') \geq v_i(A_i)$, and each $j \in N \setminus O$ satisfies $v_j\left(A_j''\right) = 0 = v_j(A_j)$. So either A'' is a Pareto improvement on A, which the offline search finds, or $v_i(A_i'') = v_i(A_i') = v_i(A_i)$ for every $i \in O$. In the latter case, each owner values the items returned to her at 0, so every returned item lies in Z, and the agent who is strictly better off in A' is some $j \in N \setminus O$; but $A_j' \subseteq Z$ gives $v_j\left(A_j'\right) \leq v_j(Z) = 0$, a contradiction. □

Proof of Proposition 4. Let $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$ be the unknown instance, and write $u(B) \coloneqq (v_i(B_i))_{i \in N} \in \mathbb{R}^N$ for $B \in \mathcal{A}_0(\mathcal{I})$. Call a vector of $\mathbb{R}^N$ *lexicographically nonpositive* if it is $\mathbf{0}$ or its first nonzero entry is negative; the sum of two such vectors is again one, whereas $u(A') - u(A)$ is nonzero and nonnegative if A' is a Pareto improvement on A.

First, suppose that $\mathcal{C}$ consists of additive instances. We claim that $A \in \mathcal{A}_0(\mathcal{I})$ is PO if every item g and the agent p who holds it in A satisfy one of the following: (i) $v_i(\{g\}) \leq 0$ for every $i < p$ and $v_p(\{g\}) > 0$; (ii) $v_p(\{g\}) \geq 0 \geq v_i(\{g\})$ for every $i \neq p$; (iii) $p = |N|$ and $v_i(\{g\}) < 0$

for every $i < p$. Indeed, for any $A' \in \mathcal{A}_0(\mathcal{I})$, let $q(g)$ be the agent who holds g in A', so that $u(A') - u(A) = \sum_{g \in M} w_g$ with $w_g \coloneqq v_{q(g)}(\{g\}) \mathbf{1}_{N|\{q(g)\}} - v_p(\{g\}) \mathbf{1}_{N|\{p\}}$, where p holds g in A. If $q \coloneqq q(g) \neq p$, then w_g is lexicographically nonpositive: in case (i), its entry at p is negative and its entry at q is nonpositive if $q < p$; in case (ii), both entries are nonpositive; and in case (iii), its entry at $q < p$ is negative. So $u(A') - u(A)$ is lexicographically nonpositive, and A' is not a Pareto improvement on A. The comparisons of $\{g\}$ with $\emptyset$ in both directions give the sign of $v_i(\{g\})$ for all $i \in N$ and $g \in M$, with $2|N||M|$ queries, and giving each item g to its *lexicographic owner*, i.e., the first agent who values g positively if there is one, otherwise the first agent who values g at 0 if there is one, and otherwise agent $|N|$, satisfies the claim.

Algorithm 14 An algorithm that finds a PO allocation in monotone additive instances by group testing.

Input: The sets of agents N and items M of an unknown monotone additive instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$.
Output: An allocation $A \in \mathrm{PO}(\mathcal{I})$.

$R \leftarrow M$.
for all $i \in \{1, 2, \ldots, |N| - 1\}$ **do**
    $A_i \leftarrow \textsc{Positive}(i, R)$ and $R \leftarrow R \setminus A_i$.
$A_{|N|} \leftarrow R$.
return A.

function $\textsc{Positive}(i, S)$ ▷ *Returns* $\{g \in S \mid v_i(\{g\}) > 0\}$.
    if $S = \emptyset$ or a comparison query verifies that $v_i(S) \leq v_i(\emptyset)$ **then**
        return $\emptyset$.
    else if $|S| = 1$ **then**
        return S.
    Split S into disjoint sets S_1 and S_2 with $|S_1| = \lceil |S|/2 \rceil$ and $|S_2| = \lfloor |S|/2 \rfloor$.
    return $\textsc{Positive}(i, S_1) \cup \textsc{Positive}(i, S_2)$.

Next, suppose that $\mathcal{C}$ consists of monotone additive instances, and let $n' \coloneqq |N|$ and $m' \coloneqq |M|$. As item values are nonnegative, the comparison of $S \subseteq M$ with $\emptyset$ by agent i tells whether S contains an item that she values positively. Algorithm 14 lets each agent $i < n'$ in turn take the set A_i of items in R_{i-1} that she values positively, where $R_0 \coloneqq M$ and $R_i \coloneqq R_{i-1} \setminus A_i$, and gives $A_{n'} \coloneqq R_{n'-1}$ to agent n'. Let $A' \in \mathcal{A}_0(\mathcal{I})$ satisfy $v_i(A'_i) \geq v_i(A_i)$ for every $i \in N$. By induction on $i < n'$, we show that $A'_i \supseteq A_i$ and $v_i(A'_i) = v_i(A_i)$: if this holds for every $j < i$, then $A'_i \subseteq M \setminus \bigcup_{j<i} A'_j \subseteq R_{i-1}$, and agent i values every item of $R_{i-1} \setminus A_i$ at 0, so $v_i(A_i) \leq v_i(A'_i) \leq v_i(R_{i-1}) = v_i(A_i)$, and agent i values $R_{i-1} \setminus A'_i$ at 0, so that $A_i \subseteq A'_i$. Then $A'_{n'} \subseteq R_{n'-1} = A_{n'}$, so $v_{n'}\left(A'_{n'}\right) \leq v_{n'}(A_{n'})$, and no agent is strictly better off in A'; hence $A \in \mathrm{PO}(\mathcal{I})$.

For the count, consider an agent $i < n'$ with $r \coloneqq |R_{i-1}| \geq 1$ and $a \coloneqq |A_i|$, and let $D \coloneqq \lceil \log_2 r \rceil$. The calls of $\textsc{Positive}$ for agent i form a binary tree whose nodes at each depth are disjoint subsets of R_{i-1}, of depth at most D, and each call makes at most one comparison; besides the root, they are the two children of every node of size at least 2, hence of depth less than D, that contains an item of A_i. At depth d, at most $\min\left\{a, 2^d\right\}$ nodes contain an item of A_i, so agent i makes one comparison if $a = 0$, and otherwise, with $k \coloneqq \lceil \log_2 a \rceil \leq D$ and $2^k < 2a$, at most

$$1 + 2 \sum_{d=0}^{D-1} \min\left\{a, 2^d\right\} \leq 1 + 2\left(2^k - 1 + a(D - k)\right) \leq 1 + 6a + 2a \log_2(r/a). \tag{28}$$

Let s be the number of agents $i < n'$ with $a_i \coloneqq |A_i| \geq 1$, so that $s \leq \min\{n', m'\}$, and let $a \coloneqq \sum_{i<n'} a_i \leq m'$. If $s = 0$, the agents make $n' - 1$ comparisons in total. Otherwise, as $r \leq m'$ for every agent, $x \log_2(m'/x) \leq m' \log_2(e)/e < m'$ for $x > 0$, and the entropy of the distribution $(a_i/a)_{i:a_i \geq 1}$ is at most $\log_2 s$, the total number of comparisons is at most

$$n' - 1 + 6m' + 2 \sum_{i:a_i \geq 1} a_i \log_2 \left(\frac{m'}{a_i} \right) = n' - 1 + 6m' + 2a \log_2 \left(\frac{m'}{a} \right) + 2 \sum_{i:a_i \geq 1} a_i \log_2 \left(\frac{a}{a_i} \right) \tag{29}$$

$$\leq n' - 1 + 8m' + 2m' \log_2 s. \tag{30}$$

This is at most $n - 1 + 10m \log_2 n$ if $n' \geq 2$, and no comparison is made if $n' = 1$; so $\mathrm{SR}_{\mathrm{det}}(\mathrm{PO}) = O(n + m \log n)$.

Finally, suppose that $\mathcal{C}$ consists of homogeneous additive instances, with $v_i = v$ for every $i \in N$. Every $A \in \mathcal{A}_0(\mathcal{I})$ satisfies $\sum_{i \in N} v(A_i) = v(M)$, whatever the signs of the item values, whereas a Pareto improvement on A would increase this sum. So every complete allocation, e.g., $(M, \emptyset, \ldots, \emptyset)$, is PO, and returning it needs no query. □

In the following proofs, let $N \coloneqq \{1, 2, \ldots, n\}$ and $M \coloneqq \{1, 2, \ldots, m\}$. As in the proof of Theorem 10, we assign the vector $\mathbf{1}_{M|X} - \mathbf{1}_{M|Y}$ to a comparison of X with Y and the vector $\mathbf{1}_{M|X}$ to a value query of X. If a set Q of queries contains fewer than m queries to an agent i, then some nonzero $d \in \mathbb{R}^M$ is orthogonal to the vectors of all of them, and replacing v_i with $v_i + td$ for any $t \in \mathbb{R}$ changes no answer to the queries in Q. For an instance $\mathcal{J} \in \mathcal{C}$ and $A \in \mathrm{PO}(\mathcal{J})$, we say that a set Q of queries is *refuted* at $(\mathcal{J}, A)$ if some $\mathcal{J}' \in \mathcal{C}$ with the same agents and items satisfies $Q \cap \Delta(\mathcal{J}, \mathcal{J}') = \emptyset$ and $A \notin \mathrm{PO}(\mathcal{J}')$. If every set of fewer than L queries is refuted at $(\mathcal{J}, A)$ for every $A \in \mathrm{PO}(\mathcal{J})$, then Lemma 16 with $\mathcal{P}' \coloneqq \mathcal{P} \coloneqq \mathrm{PO}$ gives $\mathrm{SR}_{\mathrm{rand}}(\mathrm{PO}) \geq L$; if this holds for one $A \in \mathrm{PO}(\mathcal{J})$, then the argument for verification in the proof of Theorem 10, with Lemma 11, gives $\mathrm{VR}_{\mathrm{rand}}(\mathrm{PO}) \geq L$; both hold even if value queries are allowed. Moreover, if $\emptyset \neq S \subsetneq M$ and $d \in \mathbb{R}^M$ is not constant, then some $h \in S$ and $g \in M \setminus S$ satisfy $d_g \neq d_h$, since otherwise every entry of d would equal d_{g_0} for a fixed $g_0 \in M \setminus S$. For $c \in \mathbb{R}$, let $\mathcal{J}_c$ denote the homogeneous instance with agents N, items M, and the valuations $c\mathbf{1}_{M|M}$; every complete allocation is PO at $\mathcal{J}_c$, by the proof of Proposition 4.

Lemma 25. *Let $c \in \{1, -1\}$, let $\mathcal{C}$ be the set of all monotone additive instances if $c = 1$ and of all antitone additive instances if $c = -1$, with agents N and items M, and let $A \in \mathcal{A}_0(\mathcal{J}_c)$. A set Q of queries is refuted at $(\mathcal{J}_c, A)$ if it contains fewer than m queries to some agent i such that $A_i \neq \emptyset$ if $c = 1$, and $A_i \neq M$ if $c = -1$.*

Proof. Let d be as above for agent i and $v_i \coloneqq c\mathbf{1}_{M|M}$, and let $\mathcal{J}'$ be obtained from $\mathcal{J}_c$ by replacing v_i with $v_i + td$ for some $t \in \mathbb{R}$ chosen below, so that $Q \cap \Delta(\mathcal{J}_c, \mathcal{J}') = \emptyset$. First, let $\emptyset \neq A_i \neq M$. If d is not constant, then some $h \in A_i$ and $g \in M \setminus A_i$ satisfy $d_g \neq d_h$, and $d_g > d_h$ after replacing d with $-d$ if necessary; with $t > 0$ so small that every entry of $v_i + td$ has the sign of c, so that $\mathcal{J}' \in \mathcal{C}$, swapping g and h between agent i and the holder of g, who values all items alike, is a Pareto improvement on A at $\mathcal{J}'$. If d is constant, it is a nonzero multiple of $\mathbf{1}_{M|M}$, so some t gives $v_i + td = \mathbf{0}$; at this $\mathcal{J}' \in \mathcal{C}$, moving an item of A_i to another agent if $c = 1$, or an item of another agent to agent i if $c = -1$, is a Pareto improvement on A. Finally, let $A_i = M$ if $c = 1$, and $A_i = \emptyset$ if $c = -1$; replacing d with $-d$ if necessary, cd has a negative entry, and $t \coloneqq 1/\max_{g \in M}(-cd_g)$ makes every entry of $c(v_i + td) = \mathbf{1}_{M|M} + tcd$ nonnegative and the entry of some $g^\star$ zero, so that $\mathcal{J}' \in \mathcal{C}$ and moving $g^\star$ from agent i to another agent if $c = 1$, or from its holder to agent i if $c = -1$, is a Pareto improvement on A at $\mathcal{J}'$. □

Proof of Theorem 25. Let $\varepsilon \in [0, 1/2)$, and first allow value queries. Fix $g^\star \in M$, let $\mathcal{I}_0$ be the instance with agents N, items M, and the valuation $\mathbf{0}$ for every agent, and, for $j \in N$, let $\mathcal{I}_j$ be obtained from $\mathcal{I}_0$ by giving agent j the valuation $\mathbf{1}_{M|\{g^\star\}}$; these instances are monotone additive. Every complete allocation is PO at $\mathcal{I}_0$, and a complete allocation A is PO at $\mathcal{I}_j$ if and only if $g^\star \in A_j$: if $g^\star \in A_k$ with $k \neq j$, then moving $g^\star$ to agent j makes her strictly better off and leaves everyone else indifferent, and if $g^\star \in A_j$, then every agent receives her largest possible value. So each output is valid for at most one of $\mathcal{I}_1, \dots, \mathcal{I}_n$. Since $\mathcal{I}_j$ differs from $\mathcal{I}_0$ only in the valuation of agent j, the set $\Delta(\mathcal{I}_0, \mathcal{I}_j)$ consists of queries to agent j, and these sets are pairwise disjoint. By Lemma 12 with unit weights and $a = 1$, every algorithm that performs the search correctly with probability at least $1 - \varepsilon$ makes at least $(1 - \varepsilon)\, n - 1$ queries in expectation on $\mathcal{I}_0$. Moreover, the proof of Remark 9 shows that every set of fewer than $m + n - 2$ comparisons is refuted at $(\mathcal{J}, A)$ for every $A \in \mathrm{PO}(\mathcal{J})$, where $\mathcal{J}$ is the instance there; this also holds with value queries, as the vector d there can be chosen orthogonal to the vectors $\mathbf{1}_{M|S}$ of the value queries to agent i as well, which changes no answer, so Lemma 16 gives $\mathrm{SR}_{\mathrm{rand}}(\mathrm{PO}) \geq m + n - 2$.

Now let only comparison queries be allowed. For each $B \in \Pi_n(M)$, let $\mathcal{I}_B$ be the instance with agents N, items M, and the additive valuations $v_i \coloneqq \mathbf{1}_{M|B_i}$ for $i \in N$, which are monotone. Then $\mathrm{PO}(\mathcal{I}_B) = \{B\}$: B gives every agent her largest possible value, and if a complete allocation $A \neq B$ gives some $g \in B_j$ to an agent $k \neq j$, then moving g to agent j makes her strictly better off and leaves agent k, who values g at 0, indifferent. So the n^m compatible instances $\mathcal{I}_B$ have pairwise disjoint sets of valid outputs, and Lemmas 11 and 12 give $\mathrm{SR}_{\mathrm{rand}}(\mathrm{PO}) \geq \log_2 n^m = m \log_2 n$ and $\mathrm{SR}_{\varepsilon}(\mathrm{PO}) \geq (1 - \varepsilon)\, m \log_2 n - 1$. □

Proof of Theorem 26. Let $\varepsilon \in [0, 1/2)$, and first allow value queries. Let $g^\star$, $\mathcal{I}_0$, and $\mathcal{I}_j$ be as in the proof of Theorem 25, and let $A^\circ \coloneqq (M, \emptyset, \dots, \emptyset)$, which is PO at $\mathcal{I}_0$. For each $j \in N \setminus \{1\}$, moving $g^\star$ from agent 1, who values it at 0, to agent j is a Pareto improvement on A° at $\mathcal{I}_j$. The sets $\Delta((\mathcal{I}_0, A^\circ), (\mathcal{I}_j, A^\circ))$ consist of queries to agent j and are pairwise disjoint, so Lemma 12 with unit weights shows that every algorithm that verifies correctly with probability at least $1 - \varepsilon$ makes at least $(1 - 2\varepsilon)(n - 1)$ queries in expectation on $(\mathcal{I}_0, A^\circ)$, and $\varepsilon = 0$ gives $\mathrm{VR}_{\mathrm{rand}}(\mathrm{PO}) \geq n - 1$.

Next, let $k \coloneqq \min\{n, m\} \geq 2$, and let $A \in \Pi_n(M)$ have exactly k nonempty bundles; then $A \in \mathrm{PO}(\mathcal{J}_1)$. A set of fewer than km queries contains fewer than m queries to some agent i with $A_i \neq \emptyset$, so it is refuted at $(\mathcal{J}_1, A)$ by Lemma 25, which gives $\mathrm{VR}_{\mathrm{rand}}(\mathrm{PO}) \geq km$, even if value queries are allowed.

Now let only comparison queries be allowed, and let $\ell \coloneqq \min\{n - 1, \lfloor m/2 \rfloor\} \geq 1$, $N^\star \coloneqq \{1, 2, \dots, \ell\} \subseteq N$, $F \coloneqq \{\ell + 1, \ell + 2, \dots, m\} \subseteq M$, and $f \coloneqq \ell + 1 \in N$. Let A give item i to each agent $i \in N^\star$, the $m - \ell$ items of F to agent f, and nothing to the other agents. For each $i \in N^\star$, fix an injective map $r_i : M \setminus \{i\} \to \{0, 1, \dots, m - 2\}$, let $\delta \coloneqq 2^{-m}$, and let $\mathcal{I}_0$ be the instance with agents N, items M, and the additive valuations

$$v_i \coloneqq \mathbf{1}_{M|M} - \delta \sum_{g \in M \setminus \{i\}} 2^{r_i(g)} \mathbf{1}_{M|\{g\}} \qquad \forall i \in N^\star, \tag{31}$$

$v_f \coloneqq \mathbf{1}_{M|M}$, and $v_j \coloneqq \mathbf{0}$ for the other agents j, which are monotone as $\delta 2^{m-2} < 1$. Every agent values every item at most at 1 and the items of her bundle in A at exactly 1. So, in a complete allocation A' with $v_j\left(A'_j\right) \geq v_j(A_j)$ for every $j \in N$, agent f receives at least $m - \ell$ items and each $i \in N^\star$ at least one, which accounts for all m items; hence each $i \in N^\star$ receives a single item that she values at least at 1, which is item i, and $A' = A$. Thus $A \in \mathrm{PO}(\mathcal{I}_0)$. For $(i, g) \in N^\star \times F$, let $\mathcal{I}_{i,g}$ be obtained from $\mathcal{I}_0$ by flipping the sign of the term of g in v_i, which keeps it monotone; then swapping g and i between agents i and f makes agent i, who

values g at $1+\delta 2^{r_i(g)}$, strictly better off and leaves agent f indifferent, so $A \notin \mathrm{PO}\left(\mathcal{I}_{i,g}\right)$. Since $\delta < 2^{-1-(m-2)}$, Lemma 14 applies to v_i with $c \coloneqq \mathbf{1}_{M|M}$, $U \coloneqq M \setminus \{i\}$, $\sigma_g \coloneqq -1$ for $g \in U$, and $r \coloneqq r_i$, so $\mathcal{I}_{i,g}$ answers a comparison differently from $\mathcal{I}_0$ only if it is a comparison to agent i whose normal h satisfies $t(h) = g$. So the $\ell(m-\ell)$ sets $\Delta\left(\left(\mathcal{I}_0, A\right), \left(\mathcal{I}_{i,g}, A\right)\right)$ are pairwise disjoint, and Lemma 12 with unit weights gives the bound $(1-2\varepsilon)\,\ell(m-\ell)$. Finally, if $\ell = n-1$, then $n \leq m$ and $m - \ell \geq m/2$, so $\ell(m-\ell) \geq (n-1)\,m/2 \geq nm/4$; otherwise $\ell = \lfloor m/2 \rfloor$, and $\ell(m-\ell) = \lfloor m^2/4 \rfloor \geq m^2/5 \geq m \min\{n,m\}/5$, as $m \geq 2$. □

Proof of Theorem 27. Only comparison queries are allowed. Let $o \coloneqq m$, let $K \coloneqq M \setminus \{o\}$ be the *core*, with $k \coloneqq m-1$ items, let $b \in \mathbb{R}^M$ give the items of K the distinct values $1, 2, 4, \ldots, 2^{m-2}$ and o the value 2^{m-1}, and let $\tilde{b} \in \mathbb{R}^M$ agree with b on K and with $-b$ at o. Then $\langle b, h\rangle \neq 0$ and $\langle \tilde{b}, h\rangle \neq 0$ for every $h \in \mathcal{H}_M$, as, in either inner product, the term of the item of largest value b_g in the support of h exceeds the sum of the absolute values of the other terms. For $u \in \mathbb{R}^K$, let $\bar{u} \in \mathbb{R}^M$ agree with u on K and give o the value $u(K)/2$, and, if u has positive entries, let $\delta(u) > 0$ be 2^{-m} times the least $|\langle \bar{u}, h\rangle|$ over all $h \in \mathcal{H}_M$ with $\langle \bar{u}, h\rangle \neq 0$. For $\delta \geq 0$ and $u \in \mathbb{R}^K$, let $\mathcal{G}_\delta(u)$ be the additive instance with agents N, items M, and the valuations $v_1 \coloneqq \bar{u} + \delta b$, $v_2 \coloneqq \bar{u} + \delta\tilde{b}$, and $v_j \coloneqq v_1$ for $j \geq 3$, let $\mathcal{G}_\delta^-(u)$ be the instance with the valuations $-v_i$ instead, and let $A^\circ \coloneqq (K, \{o\}, \emptyset, \ldots, \emptyset)$ and $A^- \coloneqq (\{o\}, K, \emptyset, \ldots, \emptyset)$.

Let $u \in \mathbb{R}^K$ have positive entries, and let $0 < \delta < \delta(u)$. For $i \in N$ and $h \in \mathcal{H}_M$, we have $\langle v_i, h\rangle = \langle \bar{u}, h\rangle + \delta\langle \beta, h\rangle$ with $\beta \in \left\{b, \tilde{b}\right\}$, where $|\delta\langle \beta, h\rangle| \leq \delta\left(2^m - 1\right) < 2^m\delta(u)$, so the sign of $\langle v_i, h\rangle$ is that of $\langle \bar{u}, h\rangle$ if $\langle \bar{u}, h\rangle \neq 0$, and that of $\langle \beta, h\rangle$, which is nonzero, otherwise. So $\mathcal{G}_\delta(u)$ is non-degenerate, and, as $\langle \bar{u}, \mathbf{1}_{M|\{g\}}\rangle > 0$ for every $g \in M$, its item values are positive, so it is monotone; likewise, $\mathcal{G}_\delta^-(u)$ is antitone and non-degenerate. By this *sign rule*, which holds for all $h \in \mathcal{H}_M$ at once, the answers of both instances to all comparisons are determined by the signs of $\langle \bar{u}, h\rangle$ for $h \in \mathcal{H}_M$ and do not depend on δ.

We claim that $A^\circ \in \mathrm{PO}\left(\mathcal{G}_\delta(u)\right)$, and $A^- \in \mathrm{PO}\left(\mathcal{G}_\delta^-(u)\right)$, if and only if u is not balanced. As $v_2 = v_1 - 2^m\delta\mathbf{1}_{M|\{o\}}$, every complete allocation B satisfies $\sum_{i\in N} v_i(B_i) = v_1(M) - 2^m\delta\,[o \in B_2]$. So, if A' is a Pareto improvement on A° at $\mathcal{G}_\delta(u)$, the gains $v_i\left(A_i'\right) - v_i\left(A_i^\circ\right) \geq 0$ of the agents sum to at most $2^m\delta < 2^m\delta(u) \leq \bar{u}_g < v_1(\{g\})$ for every $g \in M$, and every agent $j \geq 3$, who has the valuation v_1, receives $A_j' = \emptyset$. If $o \in A_2'$, then $A_1' \subseteq K$, and $v_1\left(A_1'\right) \geq v_1(K)$ forces $A_1' = K$, as item values are positive, so $A' = A^\circ$, a contradiction. So $o \in A_1'$, and, with $S \coloneqq A_2' \subseteq K$ and $h \coloneqq \mathbf{1}_{M|S} - \mathbf{1}_{M|\{o\}}$, the gains of agents 1 and 2 are $v_1(\{o\}) - v_1(S) = -\langle v_1, h\rangle$ and $v_2(S) - v_2(\{o\}) = \langle v_2, h\rangle$. By the sign rule, $\langle v_1, h\rangle$ and $\langle v_2, h\rangle$ have the same nonzero sign unless $\langle \bar{u}, h\rangle = 0$, so both gains are nonnegative only if $\langle \bar{u}, h\rangle = u(S) - u(K)/2$ vanishes, and then u is balanced with the halves S and $K \setminus S$. Conversely, if u is balanced with a half S, then $\langle \bar{u}, h\rangle = 0$, $\langle b, h\rangle = b(S) - 2^{m-1} < 0$, and $\langle \tilde{b}, h\rangle = b(S) + 2^{m-1} > 0$, so, by the sign rule, the allocation that gives $(K \setminus S) \cup \{o\}$ to agent 1, S to agent 2, and nothing to the others is a Pareto improvement on A° that makes agents 1 and 2 strictly better off. At $\mathcal{G}_\delta^-(u)$, every agent $j \geq 3$ values every nonempty bundle negatively, so every Pareto improvement on A^- gives her $\emptyset$. For a complete allocation B with $B_j = \emptyset$ for every $j \geq 3$, let B' be obtained from B by exchanging the bundles of agents 1 and 2, so that $(A^-)' = A^\circ$; as $v_i(B_i) + v_i\left(B_i'\right) = v_i(M) = v_i\left(A_i^-\right) + v_i\left(A_i^\circ\right)$ for $i \in \{1,2\}$, we have $-v_i(B_i) + v_i\left(A_i^-\right) = v_i\left(B_i'\right) - v_i\left(A_i^\circ\right)$. So B is a Pareto improvement on A^- at $\mathcal{G}_\delta^-(u)$ if and only if B' is one on A° at $\mathcal{G}_\delta(u)$, and, as every Pareto improvement on A° gives $\emptyset$ to the agents $j \geq 3$, the claim for A^- follows from that for A°.

Now let $\varepsilon \in [0, 1/2)$, and let Λ verify PO on $\mathcal{C}$ correctly with probability at least $1-\varepsilon$ and with at most Q queries in expectation on every input. Let $k \geq 4$, and let d, $u(z) \in \mathbb{R}^K$ for $z \in \mathbb{R}^d$, and $\mathcal{I}(z)$ be as in the proof of Lemma 17, with our core K, so that $u(z)$ has positive entries and is

balanced if and only if $z \in \mathrm{KS}_d$, and its entries, and hence those of $\overline{u(z)}$, are affine functions of z on each set of z with given signs of $z_1, \ldots, z_d$. That proof, for $\mathrm{DC}_\varepsilon(\mathrm{EF})$, uses the inputs $\mathcal{I}(z)$ and the decision algorithm only through three facts: these inputs are compatible; on each set of z with given signs of $z_1, \ldots, z_d$, the answer to each comparison at $\mathcal{I}(z)$ is determined by the sign of an affine function of z; and the algorithm tells whether $z \in \mathrm{KS}_d$ at each $\mathcal{I}(z)$, correctly with probability at least $1-\varepsilon$ and with at most Q queries in expectation. So, for every family of inputs indexed by $z \in \mathbb{R}^d$ with the first two properties and every algorithm with the third, it shows that $Q \geq ck^2$ if $k \geq k_\varepsilon$, where $c > 0$ and $k_\varepsilon \geq 4$ are as there and depend only on ε. For $z \in \mathbb{R}^d$, let $\delta_z \coloneqq \delta(u(z))/2$, and consider the input $(\mathcal{G}_{\delta_z}(u(z)), A^\circ)$, or $(\mathcal{G}^-_{\delta_z}(u(z)), A^-)$. Its instance lies in $\mathcal{C}$, so Λ, with its answer negated, tells whether $z \in \mathrm{KS}_d$ there, by the claim, correctly with probability at least $1-\varepsilon$ and with at most Q queries in expectation. By the sign rule, this single instance answers each comparison of bundles X and Y for an agent as determined by the sign of $\langle \overline{u(z)}, \mathbf{1}_{M|X} - \mathbf{1}_{M|Y} \rangle$, an affine function of z on each such set, as a zero sign is resolved by the fixed sign of $\langle \beta, \mathbf{1}_{M|X} - \mathbf{1}_{M|Y} \rangle$. Hence $Q \geq ck^2$ if $k \geq k_\varepsilon$. If $k < k_\varepsilon$, then $Q \geq 1 - 2\varepsilon$ by Lemma 12, applied to two compatible inputs at only one of which the allocation is PO. Indeed, fix $g \in K$, and let $b' \in \mathbb{R}^M$ be obtained from b by exchanging the values of g and o; every complete allocation is PO at the homogeneous instance in which every agent has the valuation b, by the proof of Proposition 4, whereas, at the instance obtained from it by giving agent 2 the valuation b', exchanging g and o between agents 1 and 2 is a Pareto improvement on A°, as $b_o > b_g$ and $b'_g > b'_o$; these instances lie in $\mathcal{C}$, and the same holds for their negations and A^-. As $(m-1)^2 \geq m^2/4$, this gives $Q \geq \min\{c/4, (1-2\varepsilon)/k_\varepsilon^2\} m^2$ for every $m \geq 2$. □

Proof of Theorem 28. Let $\varepsilon \in [0, 1/2)$, let $\mathcal{C}$ be the set of all antitone additive instances with agents N and items M, and first allow value queries. Let $A \in \mathcal{A}_0(\mathcal{J}_{-1}) = \mathrm{PO}(\mathcal{J}_{-1})$. At most one agent i has $A_i = M$, and none does if A has two nonempty bundles. So, by Lemma 25, every set of fewer than $(n-1)m$ queries is refuted at $(\mathcal{J}_{-1}, A)$, and every set of fewer than nm queries is refuted if A has two nonempty bundles, which gives $\mathrm{SR}_{\mathrm{rand}}(\mathrm{PO}) \geq (n-1)m$ and $\mathrm{VR}_{\mathrm{rand}}(\mathrm{PO}) \geq nm$.

Now let only comparison queries be allowed, and let $\delta \coloneqq 2^{-1-m}$. For search, let $\mathcal{I}_0$ be the instance with agents N, items M, and the valuation $-\delta \sum_{g \in M} 2^{g-1} \mathbf{1}_{M|\{g\}}$ for every agent, so that $\mathrm{PO}(\mathcal{I}_0) = \mathcal{A}_0(\mathcal{I}_0)$, and, for $(i, g) \in N \times M$, let $\mathcal{I}_{i,g}$ be obtained from $\mathcal{I}_0$ by replacing the value of g for agent i with 0. A complete allocation A is PO at $\mathcal{I}_{i,g}$ if and only if $g \in A_i$: if $g \in A_k$ with $k \neq i$, then moving g to agent i makes agent k strictly better off and leaves agent i indifferent, and if $g \in A_i$, then A maximizes the sum of the values of the agents at $\mathcal{I}_{i,g}$, which a Pareto improvement would increase, since every other item has the same value for all agents. So each output is valid for at most m of the nm inputs $\mathcal{I}_{i,g}$, one for each item. By Lemma 14 with $c \coloneqq \mathbf{0}$, $U \coloneqq M$, $\sigma_g \coloneqq -1$, and $r(g) \coloneqq g - 1$ for $g \in M$, replacing σ_g with 0 shows that $\mathcal{I}_{i,g}$ answers a comparison differently from $\mathcal{I}_0$ only if it is a comparison to agent i whose normal h satisfies $t(h) = g$. So these sets are pairwise disjoint, and Lemma 12 with unit weights and $a = m$ gives $\mathrm{SR}_\varepsilon(\mathrm{PO}) \geq (1-\varepsilon)nm - m$.

For verification, let $A \in \Pi_n(M)$ be arbitrary, and, for each $i \in N$, fix a bijection $r_i : M \to \{0, 1, \ldots, m-1\}$ with $r_i(g) < r_i(g')$ whenever $g \in A_i$ and $g' \notin A_i$, and let $\mathcal{I}'_0$ be the instance with agents N, items M, and the valuations $v_i \coloneqq -\delta \sum_{g \in M} 2^{r_i(g)} \mathbf{1}_{M|\{g\}}$ for $i \in N$. Every item $g' \notin A_i$ satisfies $v_i(\{g'\}) \leq -\delta 2^{|A_i|} < v_i(A_i)$, and all item values are negative, so agent i values every bundle that contains an item outside A_i below A_i. Hence a complete allocation A' with $v_i(A'_i) \geq v_i(A_i)$ for every $i \in N$ satisfies $A'_i \subseteq A_i$ for every $i \in N$, so $A' = A$, and $A \in \mathrm{PO}(\mathcal{I}'_0)$. For $i \in N$ and $g \in M \setminus A_i$, let $\mathcal{I}'_{i,g}$ be obtained from $\mathcal{I}'_0$ by replacing the value of g for agent i with 0; then moving g from its holder to agent i is a Pareto improvement on A at $\mathcal{I}'_{i,g}$. By Lemma 14 as above, with $r \coloneqq r_i$, the $(n-1)m$ sets $\Delta\left((\mathcal{I}'_0, A), \left(\mathcal{I}'_{i,g}, A\right)\right)$ are pairwise disjoint, and Lemma 12

with unit weights gives $\mathrm{VR}_\varepsilon(\mathrm{PO}) \geq (1-2\varepsilon)\,(n-1)\,m$.

Finally, let $\mathcal{C}$ be the set of all additive instances with agents N and items M, and allow value queries again. Let $A \in \mathcal{A}_0\,(\mathcal{J}_0) = \mathrm{PO}\,(\mathcal{J}_0)$, let Q contain fewer than m queries to an agent i, and let d be as above. For some $g \in M$, $d_g \neq 0$, and, replacing d with $-d$ if necessary, $d_g < 0$ if $g \in A_i$ and $d_g > 0$ otherwise. If agent i has the valuation d instead of $\mathbf{0}$, then moving g from A_i to another agent, or to agent i from its holder, makes agent i strictly better off and leaves everyone else at 0, so Q is refuted at $(\mathcal{J}_0, A)$. So every set of fewer than nm queries is refuted at $(\mathcal{J}_0, A)$ for every $A \in \mathrm{PO}\,(\mathcal{J}_0)$, which gives $\mathrm{SR}_{\mathrm{rand}}(\mathrm{PO}) \geq nm$ and $\mathrm{VR}_{\mathrm{rand}}(\mathrm{PO}) \geq nm$. An algorithm that is correct on all additive instances is correct on the antitone ones, so the bounds on $\mathrm{SR}_\varepsilon(\mathrm{PO})$ and $\mathrm{VR}_\varepsilon(\mathrm{PO})$ hold as well. □

Proof of Remark 9. We identify a set of comparisons with the set of coordinates that they read (Section 2.1). For the upper bounds, let $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$, and let $R_0 \coloneqq M$. On monotone additive instances, let A be the output of Algorithm 14, with A_i and R_i as in the proof of Proposition 4. The comparisons of $\{g\}$ with $\emptyset$ by agent i for $i < n$ and $g \in A_i$, and of R_i with $\emptyset$ by agent i for $i < n$ with $R_i \neq \emptyset$, certify A: at every instance in $\mathcal{C}$ that answers them alike, each agent $i < n$ values every item of A_i positively and R_i at 0, so A_i is the set of items of R_{i-1} that she values positively, and the proof of Proposition 4 shows that A is PO there. If $R_{n-1} \neq \emptyset$, these are at most $m-1$ comparisons of the first kind and $n-1$ of the second; otherwise, with the least t such that $R_t = \emptyset$, they are m comparisons of the first kind and $t-1 \leq n-2$ of the second. On antitone additive instances, let $A_i \coloneqq \{g \in R_{i-1} \mid v_i\,(\{g\}) = 0\}$ and $R_i \coloneqq R_{i-1} \setminus A_i$ for $i < n$, and $A_n \coloneqq R_{n-1}$. The at most $(n-1)\,m$ comparisons of $\emptyset$ with $\{g\}$ by agent i for $i < n$ and $g \in R_{i-1}$ certify A: at every instance in $\mathcal{C}$ that answers them alike, each agent $i < n$ values A_i at 0, her largest possible value, and every item of A_n negatively, so in a complete allocation in which every agent is weakly better off, each agent $i < n$ receives only items that she values at 0, agent n receives every item of A_n, and nobody is strictly better off. On additive instances, let A give each item g to its lexicographic owner p, as in the proof of Proposition 4, and, for a case among (i)–(iii) there that holds, take the comparisons of $\{g\}$ with $\emptyset$ by the agents $i \leq p$ in case (i); those of $\emptyset$ with $\{g\}$ by agent p and of $\{g\}$ with $\emptyset$ by every agent $i \neq p$ in case (ii); and those of $\emptyset$ with $\{g\}$ by every agent $i < p$ in case (iii). These are at most nm comparisons, and at every additive instance that answers them alike, each item and its holder satisfy the same case, so A is PO there by the claim in the proof of Proposition 4.

For the lower bound on antitone additive instances, the proof of Theorem 28 shows that every set of fewer than $(n-1)\,m$ comparisons is refuted at $(\mathcal{J}_{-1}, A)$ for every $A \in \mathrm{PO}\,(\mathcal{J}_{-1})$, and the same holds for nm at $\mathcal{J}_0$ on additive instances. On monotone additive instances, let $z \coloneqq m$, $K \coloneqq M \setminus \{z\}$, and let $\mathcal{J}$ be the homogeneous instance with the valuations $\mathbf{1}_{M|K}$, at which every complete allocation is PO. Let $A \in \mathcal{A}_0\,(\mathcal{J})$, let o be the agent who holds z, and let $s \geq 1$ be the number of agents who hold an item of K. We show that a set Q of comparisons is refuted at $(\mathcal{J}, A)$ if it contains fewer than m comparisons to an agent $i \neq o$ with $A_i \cap K \neq \emptyset$, fewer than $m-1$ comparisons to o while $A_o \cap K \neq \emptyset$, or none to an agent $j \neq o$ with $A_j = \emptyset$. These requirements add up to $n-1+s\,(m-1) \geq m+n-2$, both if $A_o \cap K \neq \emptyset$, as $(s-1)\,m+m-1+n-s$, and otherwise, as $sm+n-s-1$; so every set of fewer than $m+n-2$ comparisons is refuted at $(\mathcal{J}, A)$ for every A. If no comparison is made to an agent $j \neq o$ with $A_j = \emptyset$, then giving her the valuation $\mathbf{1}_{M|M}$ makes moving z from agent o, who values it at 0, to agent j a Pareto improvement on A. Otherwise, let i be an agent with $A_i \cap K \neq \emptyset$, and let $d \in \mathbb{R}^M \setminus \{\mathbf{0}\}$ be orthogonal to the vectors of the comparisons in Q to agent i; if $i = o$, there are fewer than $m-1$ of them, and d can be chosen with $d_z = 0$, since their restrictions to K do not span $\mathbb{R}^K$. As in the proof of Lemma 25, replace v_i with $v_i + td$ for some $t \in \mathbb{R}$ that keeps every item value nonnegative. If $d_z \neq 0$, so that $i \neq o$, then,

with $d_z > 0$ after replacing d with $-d$ if necessary and a small $t > 0$, moving z from agent o to agent i is a Pareto improvement on A. If $d_z = 0$ and $A_i \not\supseteq K$, the restriction of d to K is nonzero, and Lemma 25's argument within K applies: if it is not constant, swapping some $g \in K \setminus A_i$ and $h \in A_i \cap K$ with $d_g > d_h$ between agent i and the holder of g is a Pareto improvement on A for a small $t > 0$, and if it is constant, some t gives $v_i + td = \mathbf{0}$, and moving an item of $A_i \cap K$ to another agent is one. If $d_z = 0$ and $A_i \supseteq K$, then, with a negative entry of d on K after replacing d with $-d$ if necessary and $t \coloneqq 1/\max_{g \in K}(-d_g)$, agent i values some $g^\star \in K \cap A_i$ at 0, and moving it to another agent is a Pareto improvement on A.

For the second statement, let $\mathcal{I}_0 \in \mathcal{C}$ be monotone additive, and let a set Q_0 of at most $m+n-2$ comparisons certify some $A_0 \in \mathrm{PO}(\mathcal{I}_0)$. The deterministic algorithm that makes the comparisons of Q_0, returns A_0 if their answers agree with $\mathcal{I}_0$, and runs Algorithm 14 otherwise is correct on every instance in $\mathcal{C}$ and makes at most $m + n - 2$ comparisons on $\mathcal{I}_0$.

For the last statement, let A' be a Pareto improvement on a complete allocation A that moves the fewest items, and consider the multigraph on N with an edge between the agents who hold g in A and in A', for each item g that they hold differently. Suppose that some agent ℓ is incident to exactly one edge, for an item g that moves between ℓ and an agent j, and let A^g be obtained from A by moving only g, and A^{-g} from A' by moving g back. If ℓ gives g to j, then $v_\ell(\{g\}) \leq 0$, as ℓ only loses g; so A^g is a Pareto improvement on A if $v_j(\{g\}) \geq 0$ and one of the two values is nonzero, and otherwise $v_j(\{g\}) < 0$ or both values are 0, and A^{-g} is one, as ℓ returns to $v_\ell(A_\ell)$, j weakly gains, strictly if $v_j(\{g\}) < 0$, and every other agent keeps her value in A', so that, if both values are 0, some agent, possibly j, is strictly better off than in A, as in A'. Symmetrically, if ℓ receives g from j, then $v_\ell(\{g\}) \geq 0$, and A^g is a Pareto improvement on A if $v_j(\{g\}) \leq 0$ and one of the two values is nonzero, and otherwise A^{-g} is one. By the choice of A', either A^g is a Pareto improvement on A that changes the bundles of two agents only, or no such ℓ exists; then every agent with an edge has at least two, so the agents whose bundles change are at most as many as the moved items, and at most $\min\{n, m\}$, as $m \geq 2$. The comparisons of A'_j with A_j by these agents j, or of A^g_j with A_j, in the direction that shows the strict gain for one strictly better off agent and the weak gain for the others, show that the allocation is a Pareto improvement on A in every instance that answers them alike. □

B.5 Section 3.5

Proof of Theorem 29. Suppose that $\mathcal{C}$ consists of additive instances; as noted at the beginning of Appendix B, we may assume that it contains every additive instance with at most n agents and at most m items, so that, for the agents and items of the unknown instance, which the algorithm is given, it is of the form required by Theorem 1, with $\mathbb{R}^M$ as every domain. Let $\mathcal{P} \in \{\mathrm{XEF1}, \mathrm{IEF1}\}$, and let $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$ be the unknown instance. First, for each agent $i \in N$, find an item g_i that maximizes $v_i(\{g_i\})$ by comparing the singletons in turn, with $|M| - 1$ comparisons, and apply Theorem 2 to v_i with the pivots $\emptyset$ and $\{g_i\}$, which gives the signs of $v_i(T)$ and of $v_i(T) - v_i(\{g_i\})$ for every $T \subseteq M$; these are $O\left(nm^2 \log m\right)$ comparisons in total. The instances of $\mathcal{C}$ with agents N and items M that answer all of them as $\mathcal{I}$ does are the additive instances with these agents and items in which the valuation function of each agent i lies in the intersection K_i of the closed and open halfspaces of $\mathbb{R}^M$ that the answers of agent i prescribe. Each K_i is convex, so the set $\mathcal{C}'$ of these instances is of the form required by Theorem 1, and $(\mathcal{P} \cap \mathrm{PO})(\mathcal{I}') \neq \emptyset$ for every $\mathcal{I}' \in \mathcal{C}'$ [24].

We bound the certificate size of $\mathcal{P} \cap \mathrm{PO}$ on $\mathcal{C}'$. Let $\mathcal{I}' = \left(N, M, (v'_i)_{i \in N}\right) \in \mathcal{C}'$, let $A \in (\mathcal{P} \cap \mathrm{PO})(\mathcal{I}')$, and let $O \coloneqq \{i \in N \mid A_i \neq \emptyset\}$, so that $|O| \leq \min\{|N|, |M|\}$. For each $i \in O$, let Q_i consist of the comparisons that Theorem 2 makes at $\mathcal{I}'$ for v'_i with the pivot A_i, and of the

$O(|M|)$ comparisons to agent i by which the verifier in the proof of Proposition 3 decides at $\mathcal{I}'$ whether agent i satisfies the condition of $\mathcal{P}$ toward every agent. Let $\mathcal{I}'' = \left(N, M, (v_i'')_{i \in N}\right) \in \mathcal{C}'$ answer the comparisons of $\bigcup_{i \in O} Q_i$ as $\mathcal{I}'$ does. Then each $i \in O$ satisfies the condition of $\mathcal{P}$ at $\mathcal{I}''$, and every $i \in N$ has the same signs of $v_i''(T) - v_i''(A_i)$ for $T \subseteq M$ as at $\mathcal{I}'$: for $i \in O$ by Q_i, and for $i \notin O$, for whom $A_i = \emptyset$, as $\mathcal{I}'$ and $\mathcal{I}''$ lie in $\mathcal{C}'$. These signs determine whether some complete allocation is a Pareto improvement on A, so $A \in \mathrm{PO}(\mathcal{I}'')$. An agent $i \notin O$ satisfies the condition of XEF1 if and only if, for each $j \in N$, some $R \in \binom{M}{\leq 1}$ has $v_i''(A_j \triangle R) \leq 0$, and that of IEF1 if and only if $v_i''(A_j) \leq \max\{0, v_i''(\{g_i\})\}$ for each $j \in N$, as g_i maximizes $v_i''(\{g\})$ over $g \in M$ by the comparisons that found it; these signs are the same at all instances of $\mathcal{C}'$, so the condition holds at $\mathcal{I}''$ as it does at $\mathcal{I}'$. Hence $\bigcup_{i \in O} Q_i$, which has $O\left(\min\{|N|, |M|\} |M|^2 \log |M|\right)$ comparisons, certifies A at $\mathcal{I}'$, and $E_\emptyset(\mathcal{P} \cap \mathrm{PO}) = 0$ on $\mathcal{C}'$. So Theorem 1 with $E' = O\left(km^2 \log m\right)$, where $k \coloneqq \min\{n, m\}$, solves the search on $\mathcal{C}'$ with $O\left(km^2 \log m \cdot nm^2 / \log(km)\right)$ further comparisons, as $x / \log x$ increases for $x \geq e$.

Finally, we show the inclusion $\mathrm{EFX}_0(\mathcal{I}) \subseteq \mathrm{XEF1}(\mathcal{I}) \cap \mathrm{IEF1}(\mathcal{I})$ stated after the theorem, together with $\mathrm{EFX}_0(\mathcal{I}) \subseteq \mathrm{EF1}(\mathcal{I})$, for every additive instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$. Let $A \in \mathrm{EFX}_0(\mathcal{I})$, which is complete. For every pair (i, j) with $v_i(A_i) \geq v_i(A_j)$, $R \coloneqq \emptyset$ satisfies the conditions of EF1, XEF1, and IEF1 in Definitions 6 and 8. If $v_i(A_i) < v_i(A_j)$, then $i \neq j$, and A_j contains an item g with $v_i(\{g\}) > 0$, or A_i contains one with $v_i(\{g\}) < 0$, as otherwise $v_i(A_i) \geq 0 \geq v_i(A_j)$. In the first case, $g \in M_{i,j}^+$, and Definition 7 gives $v_i(A_i) \geq v_i(A_j) - v_i(\{g\})$; in the second, $g \in M_i^-$, and it gives $v_i(A_i) - v_i(\{g\}) \geq v_i(A_j)$. As $A_i \cap A_j = \emptyset$ and v_i is additive, either inequality means that $R \coloneqq \{g\}$ satisfies the conditions of all three properties for (i, j). □

Proof of Remark 10. Let $\mathcal{C}$ consist of antitone additive instances, each with at least as many agents as items, and let $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$ be the unknown instance, so that $|M| \leq |N|$. As v_i is antitone, the comparison of whether $v_i(\emptyset) \leq v_i(\{g\})$ tells whether $v_i(\{g\}) = 0$; these $|N||M|$ comparisons give the set Z of items that some agent values at 0, and, for each $g \in Z$, we fix an agent s_g with $v_{s_g}(\{g\}) = 0$. Every agent values every item of $P \coloneqq M \setminus Z$ negatively, and $p \coloneqq |P| \leq |N|$. Fix p agents $a_1, a_2, \ldots, a_p$, let $S \coloneqq \{a_1, a_2, \ldots, a_p\}$, and start with the family of all bijections from S to P. For $t = 1, 2, \ldots, p$ in turn, find, among the items that a_t receives in some remaining bijection, one of largest value to a_t and, by comparing it with each of the others, all items of the same value, with fewer than $2p$ comparisons, and keep the bijections that give a_t one of these items. Let μ be a remaining bijection, and let $A_i \coloneqq \{g \in Z \mid s_g = i\}$, together with $\mu(i)$ if $i \in S$, for each $i \in N$. This makes at most $|N||M| + 2p^2 \leq 3|N||M| \leq 3nm$ comparisons.

No bijection μ' from S to P gives every agent of S at least her value in μ and some agent more: otherwise, let t be the least index with $v_{a_t}(\{\mu'(a_t)\}) \neq v_{a_t}(\{\mu(a_t)\})$; then μ' gives each $a_{t'}$ with $t' < t$ an item of the largest value at step t', so it remains at step t, and $v_{a_t}(\{\mu'(a_t)\}) \leq v_{a_t}(\{\mu(a_t)\})$, a contradiction.

The allocation A is EF1: each agent i values $A_i \setminus R$ at 0 for $R \coloneqq A_i \cap P$, which is $\{\mu(i)\}$ or $\emptyset$, and every other bundle at most at 0. For PO, let $C \in \mathcal{A}_0(\mathcal{I})$ satisfy $v_i(C_i) \geq v_i(A_i)$ for every $i \in N$. Moving each $g \in Z$ from its holder in C to s_g, who values it at 0, decreases no value, as no item has positive value, so we may assume that $C_i \cap Z = A_i \cap Z$ for every $i \in N$ and show that no agent is strictly better off in C than in A. Every agent $i \notin S$ then has $v_i(C_i) \geq v_i(A_i) = 0$, so $C_i \cap P = \emptyset$, and she is not strictly better off. Hence the sets $D_i \coloneqq C_i \cap P$ for $i \in S$ partition P, with $v_i(D_i) = v_i(C_i) \geq v_i(A_i) = v_i(\{\mu(i)\})$; suppose that some agent of S is strictly better off in D than in μ. If some $i_0 \in S$ has $D_{i_0} = \emptyset$, let i_{r+1} be the agent with $\mu(i_r) \in D_{i_{r+1}}$ for $r = 0, 1, \ldots$, until some i_t holds at least two items. This walk visits no agent twice, as i_0 holds no item and each i_r with $1 \leq r < t$ holds only $\mu(i_{r-1})$, so that it is reached only from i_{r-1}; hence it ends. Returning $\mu(i_r)$ to i_r for $0 \leq r < t$ gives $i_0, \ldots, i_{t-1}$ their values in μ, makes i_t, who loses the item

$\mu(i_{t-1})$ of negative value, strictly better off than in D, changes no other bundle, and leaves one agent fewer with an empty bundle. Repeating this yields a partition of P in which every agent of S holds one item, that is, a bijection from S to P that gives every agent of S at least her value in μ and some agent more, a contradiction. Hence $A \in \mathrm{EF1}(\mathcal{I}) \cap \mathrm{PO}(\mathcal{I})$. The algorithm and this proof use $|M| \leq |N|$ only through $p \leq |N|$, so they also work, with at most $|N||M| + 2p^2$ comparisons, whenever at most $|N|$ items have negative value to every agent. □

Proof of Theorem 30. Let $N \coloneqq \{1, 2, \ldots, n\}$ and $M \coloneqq \{1, 2, \ldots, m\}$, let $\varepsilon \in [0, 1/2)$, and let $\mathcal{J}_c$ and refuted sets of queries be as before Lemma 25. We first collect facts about the four properties EF1, XEF1, IEF1, and EFX_0. For every additive instance $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$, every $A \in \mathrm{EF1}(\mathcal{I}) \cup \mathrm{XEF1}(\mathcal{I}) \cup \mathrm{IEF1}(\mathcal{I})$, and hence, by the proof of Theorem 29, every $A \in \mathrm{EFX}_0(\mathcal{I})$, satisfies

$$v_i(A_j) - v_i(A_i) \leq \max_{g \in M} |v_i(\{g\})| \qquad \forall i, j \in N, \tag{32}$$

as, for $i \neq j$ and the set $R \in \binom{M}{\leq 1}$ of Definitions 6 and 8, removing R from both A_i and A_j, which are disjoint, or toggling it in one of them changes $v_i(A_j) - v_i(A_i)$ by at most $|v_i(R)|$. If $\mathcal{I}$ is monotone, then each of the four properties contains $\mathrm{EF}(\mathcal{I})$, with $R \coloneqq \emptyset$ for the first three, and as $v_i(A_j \setminus \{g\}) \leq v_i(A_j) \leq v_i(A_i)$ and $M_i^- = \emptyset$ for EFX_0. At $\mathcal{J}_c$ with $c \in \{1, -1\}$, Eq. (32) means that the bundle sizes of A differ by at most one. Conversely, every complete allocation A whose bundle sizes differ by at most one is PO at $\mathcal{J}_c$ and has the four properties there: if agent i envies agent j, then removing any item g of the larger of A_i and A_j from it, or adding g to the smaller one, leaves no envy, which gives the conditions of EF1, XEF1, and IEF1 with $R \coloneqq \{g\}$ and those of EFX_0; otherwise, $R \coloneqq \emptyset$ works, and removing an item from A_j if $c = 1$, or from A_i if $c = -1$, creates no envy.

First, let $\mathcal{C}$ be the set of all monotone additive instances with agents N and items M, and allow value queries. Let $A \in \mathcal{P}(\mathcal{J}_1)$, which exists by the above. As the bundle sizes of A differ by at most one, exactly r bundles are nonempty: all of them if $n \leq m$, and m singletons otherwise. So a set of fewer than mr queries contains fewer than m queries to an agent i with $A_i \neq \emptyset$, and it is refuted at $(\mathcal{J}_1, A)$ by Lemma 25. As $\emptyset \neq \mathcal{P}(\mathcal{J}_1) \subseteq \mathrm{PO}(\mathcal{J}_1)$, Lemma 16, with PO in place of $\mathcal{P}$ and $\mathcal{P}$ in place of $\mathcal{P}'$, gives $\mathrm{SR}_{\mathrm{rand}}(\mathcal{P}) \geq mr$. Next, let $g^\star$, $\mathcal{I}_0$, and $\mathcal{I}_j$ for $j \in N$ be as in the proof of Theorem 25. At $\mathcal{I}_j$, every complete allocation A with $g^\star \in A_j$ is EF, as agent j values A_j at 1 and every other bundle at 0, and every other agent values every bundle at 0; so $\mathcal{P}(\mathcal{I}_j) = \mathrm{PO}(\mathcal{I}_j)$ by the above. Likewise, at the instance $\mathcal{I}_B$ of that proof for $B \in \Pi_n(M)$, the allocation B is EF, as each agent i values B_i at $|B_i|$ and every other bundle at 0, so $\mathcal{P}(\mathcal{I}_B) = \mathrm{PO}(\mathcal{I}_B) = \{B\}$. Hence the arguments of that proof, with Lemma 16 for the counting bounds, give $\mathrm{SR}_{\mathrm{rand}}(\mathcal{P}) \geq \max\{n - 1, m \log_2 n\}$ and $\mathrm{SR}_\varepsilon(\mathcal{P}) \geq \max\{(1 - \varepsilon) n - 1, (1 - \varepsilon) m \log_2 n - 1\}$, where the bounds $n - 1$ and $(1 - \varepsilon) n - 1$ hold even if value queries are allowed.

Now let only comparison queries be allowed, and let $q \coloneqq \lfloor m/r \rfloor \geq 1$, $L \coloneqq rq \leq m$, $M^\star \coloneqq \{1, 2, \ldots, L\}$, $D_a \coloneqq \{(a-1)q + 1, \ldots, aq\}$ for $a \in \{1, 2, \ldots, r\}$, $H \coloneqq \{1, 2, \ldots, \lfloor r/2 \rfloor\}$, and $F \coloneqq \{\lfloor r/2 \rfloor + 1, \ldots, r\}$, so that $|H| = \lfloor r/2 \rfloor \geq 1$ and $|F| = \lceil r/2 \rceil$. For each $i \in H$, fix an injective map $\rho_i : M^\star \setminus D_i \to \{0, 1, \ldots, m - 1\}$, let $\delta \coloneqq 2^{-m-3}/m$, and let $\mathcal{I}^\circ$ be the instance with agents N, items M, and the additive valuations

$$v_i \coloneqq \mathbf{1}_{M|M^\star} - \delta \sum_{g \in M^\star \setminus D_i} 2^{\rho_i(g)} \mathbf{1}_{M|\{g\}} \qquad \forall i \in H, \tag{33}$$

$v_f \coloneqq \mathbf{1}_{M|M^\star}$ for $f \in F$, and $v_j \coloneqq \mathbf{0}$ for $j > r$. Each agent $a \leq r$ values every item of $M^\star$ at most at 1 and at least at $1 - \delta 2^{m-1} = 1 - 1/(16m)$, and the other items at 0, so $\mathcal{I}^\circ$ is monotone. Let B° give D_a to each agent $a \leq r$, the items of $M \setminus M^\star$ to agent 1 as well, and $\emptyset$ to the other agents.

Then $B^\circ \in \mathrm{EF}(\mathcal{I}^\circ)$, as each agent of H values her own bundle at q and every other one at most at q, those of F value every bundle of an agent $a \leq r$ at q, and the others value every bundle at 0; and $B^\circ \in \mathrm{PO}(\mathcal{I}^\circ)$, as the values of the agents sum to L, the largest possible sum, which a Pareto improvement would exceed. So $B^\circ \in \mathcal{P}(\mathcal{I}^\circ)$.

We claim that every $B \in \mathcal{P}(\mathcal{I}^\circ)$ satisfies $B_i \cap M^\star = D_i$ for every $i \in H$, so that the items of $D_F \coloneqq \bigcup_{f \in F} D_f$ belong to agents of F. First, no agent $j > r$ holds an item of $M^\star$, as moving it to agent 1 would be a Pareto improvement. Next, for $a, a' \leq r$, let $b_a \coloneqq |B_a \cap M^\star|$; by Eq. (32), $b_{a'}(1 - 1/(16m)) \leq v_a(B_{a'}) \leq v_a(B_a) + 1 \leq b_a + 1$, so $b_{a'} < b_a + 2$, as $b_{a'} \leq m$. So these numbers differ by at most one and sum to $L = rq$, whence $b_a = q$ for every $a \leq r$. Finally, let B' give $D_a \cup (B_a \setminus M^\star)$ to each agent $a \leq r$ and B_j to each agent $j > r$. The agents of F and those $j > r$ have the same values in B and B', whereas each $i \in H$ values B'_i at q and B_i at most at q, and strictly less if $B_i \cap M^\star \neq D_i$, as B_i then contains an item of $M^\star \setminus D_i$ among its q items of $M^\star$; so $B \in \mathrm{PO}(\mathcal{I}^\circ)$ gives the claim.

For $(i, g) \in H \times D_F$, let $\mathcal{I}_{i,g}$ be obtained from $\mathcal{I}^\circ$ by flipping the sign of the term of g in v_i, so that agent i values g at $1 + \delta 2^{\rho_i(g)}$; it is monotone. For every $B \in \mathcal{P}(\mathcal{I}^\circ)$, agent i holds D_i and some $f \in F$ holds g, so exchanging g and an item of D_i between agents i and f makes agent i strictly better off at $\mathcal{I}_{i,g}$ and leaves agent f, who values both items at 1, and everyone else indifferent; thus $B \notin \mathrm{PO}(\mathcal{I}_{i,g}) \supseteq \mathcal{P}(\mathcal{I}_{i,g})$. Since $\delta < 2^{-1-(m-1)}$, Lemma 14 applies to v_i with $c \coloneqq \mathbf{1}_{M|M^\star}$, $U \coloneqq M^\star \setminus D_i$, $\sigma_g \coloneqq -1$ for $g \in U$, and ρ_i in place of r, so $\mathcal{I}_{i,g}$ answers a comparison differently from $\mathcal{I}^\circ$ only if it is a comparison to agent i whose normal h satisfies $t(h) = g$. So the $\lfloor r/2 \rfloor \lceil r/2 \rceil q$ sets $\Delta(\mathcal{I}^\circ, \mathcal{I}_{i,g})$ are pairwise disjoint, and, as no output valid for $\mathcal{I}^\circ$ is valid for any $\mathcal{I}_{i,g}$, Lemma 12 with unit weights gives $\mathrm{SR}_\varepsilon(\mathcal{P}) \geq (1 - 2\varepsilon) \lfloor r/2 \rfloor \lceil r/2 \rceil q \geq (1 - 2\varepsilon) mr/8$, as $r \geq 2$ and $m \leq (q+1)r - 1$ give $mr \leq (q+1)r^2 - r \leq 2q\left(r^2 - 1\right) \leq 8 \lfloor r/2 \rfloor \lceil r/2 \rceil q$. The same sets separate the input $(\mathcal{I}^\circ, B^\circ)$ from the inputs $(\mathcal{I}_{i,g}, B^\circ)$, at which $B^\circ \notin \mathcal{P}(\mathcal{I}_{i,g})$, so Lemma 12 gives the same bound on $\mathrm{VR}_\varepsilon(\mathcal{P})$.

Lastly, let $\mathcal{C}$ be the set of all antitone additive instances with agents N and items M, and allow value queries again. Let $A \in \mathcal{P}(\mathcal{J}_{-1})$, which exists by the above. As the bundle sizes of A differ by at most one, no bundle of A is M, so every set of fewer than nm queries is refuted at $(\mathcal{J}_{-1}, A)$ by Lemma 25; Lemma 16 gives $\mathrm{SR}_{\mathrm{rand}}(\mathcal{P}) \geq nm$. Finally, let only comparison queries be allowed, and let $\mathcal{I}_0$ and $\mathcal{I}_{i,g}$ be as in the proof of Theorem 28. Each set $\mathcal{P}(\mathcal{I}_{i,g})$ is nonempty, as $\mathrm{XEF1} \cap \mathrm{PO}$ and $\mathrm{IEF1} \cap \mathrm{PO}$ are nonempty on every additive instance [24], $\mathrm{EF1} = \mathrm{IEF1}$ on antitone instances by the proof of Proposition 3, and $\mathrm{EFX}_0 \cap \mathrm{PO}$ is nonempty if $n = 2$ by Lemma 5; and its members give g to i, as those of $\mathrm{PO}(\mathcal{I}_{i,g})$ do. So the argument of that proof gives $\mathrm{SR}_\varepsilon(\mathcal{P}) \geq (1 - \varepsilon) nm - m$.

Finally, let $\mathcal{C}$ be the set of all monotone additive instances with agents N and items M again, and let $\mathcal{P}'$ be any predicate on $\mathcal{C}$ with $(\mathrm{EF} \cap \mathrm{PO})(\mathcal{I}) \subseteq \mathcal{P}'(\mathcal{I}) \subseteq \mathrm{PO}(\mathcal{I})$ for every $\mathcal{I} \in \mathcal{C}$; the search may report that $\mathcal{P}'(\mathcal{I}) = \emptyset$ at other instances. The arguments for the terms $n-1$ and $m \log_2 n$ above use only that $\mathcal{P}(\mathcal{I}_j) \subseteq \mathrm{PO}(\mathcal{I}_j)$, that $\mathcal{P}(\mathcal{I}_B) \subseteq \mathrm{PO}(\mathcal{I}_B) = \{B\}$, and that these sets are nonempty, which hold for $\mathcal{P}'$ as well, as $\mathcal{I}_j$ and $\mathcal{I}_B$ have EF + PO allocations. If moreover $\mathcal{P}'(\mathcal{I}) \subseteq \mathrm{EF1}(\mathcal{I})$ for every $\mathcal{I} \in \mathcal{C}$, then every $B \in \mathcal{P}'(\mathcal{I}^\circ)$ satisfies Eq. (32), and $B^\circ \in (\mathrm{EF} \cap \mathrm{PO})(\mathcal{I}^\circ) \subseteq \mathcal{P}'(\mathcal{I}^\circ)$, so the argument for the last term applies to $\mathcal{P}'$, for search and for verification. This covers $\mathrm{EFL} \cap \mathrm{PO}$ and $\mathrm{EFX}_0 \cap \mathrm{PO}$, as $\mathrm{EF}(\mathcal{I}) \subseteq \mathrm{EFX}_0(\mathcal{I}) \subseteq \mathrm{EF1}(\mathcal{I})$, by the above and the proof of Theorem 29, and $\mathrm{EF}(\mathcal{I}) \subseteq \mathrm{EFL}(\mathcal{I}) \subseteq \mathrm{EF1}(\mathcal{I})$, by Definitions 6 and 7, for every monotone $\mathcal{I}$. □

Proof of Remark 11. By the proof of Theorem 30, $A \in \mathcal{P}(\mathcal{J}_c)$ for $c \in \{1, -1\}$ if and only if the bundle sizes of A differ by at most one; then exactly r bundles are nonempty, and none is M. For the lower bounds, let $A \in \mathcal{P}(\mathcal{J}_c)$; the proof of Lemma 25 applies to certificates as to sets of queries, so every certificate of A at $\mathcal{J}_c$ has at least m coordinates of each of the r agents with nonempty bundles if $c = 1$, and of every agent if $c = -1$. At $\mathcal{J}_0$, let Q be a certificate of some $A \in \mathcal{P}(\mathcal{J}_0)$, and let j be an agent with no coordinate in Q and $A_j \neq M$; replacing her valuation with $\mathbf{1}_{M|\{g\}}$

for some $g \notin A_j$ gives a monotone additive instance that agrees with $\mathcal{J}_0$ on Q and at which moving g to agent j is a Pareto improvement on A, which is impossible; as at most one agent holds M, Q has coordinates of at least $n-1$ agents.

For the upper bounds, let $c \in \{1,-1\}$ and $A \in \mathcal{P}(\mathcal{J}_c)$. For each agent i with $A_i \neq \emptyset$, fix $g_i \in A_i$ and take the m coordinates of agent i with the normals $\mathbf{1}_{M|\{g_i\}} - \mathbf{1}_{M|\{g\}}$ for $g \in A_i \setminus \{g_i\}$, $\mathbf{1}_{M|\{g\}} - \mathbf{1}_{M|\{g_i\}}$ for $g \in M \setminus A_i$, and $c\mathbf{1}_{M|\{g_i\}}$; if $c=-1$, also take, for each agent i with $A_i = \emptyset$, the m coordinates with the normals $-\mathbf{1}_{M|\{g\}}$ for $g \in M$. Let $\mathcal{I}' = \left(N, M, (v'_i)_{i\in N}\right)$ be an instance of the class that agrees with $\mathcal{J}_c$ on them. Then each agent i with $A_i \neq \emptyset$ has $cv'_i(\{g_i\}) > 0$, values each item of A_i at least $v'_i(\{g_i\})$ and each other item at most $v'_i(\{g_i\})$, and so satisfies $v'_i(X) - v'_i(A_i) \leq (|X| - |A_i|)\, v'_i(\{g_i\})$ for every $X \subseteq M$; if $c=-1$, every agent with an empty bundle values every item negatively. Recall that the bundle sizes of A differ by at most one. If $c=1$, an agent i with $A_i \neq \emptyset$ values each other bundle without one of its items, or an empty one, at most $|A_i| v'_i(\{g_i\}) \leq v'_i(A_i)$, and an agent with an empty bundle, who exists only if $n > m$, satisfies the conditions of EF1 because every bundle has at most one item. If $c=-1$, an agent i with $A_i \neq \emptyset$ values $A_i \setminus \{g_i\}$ at least $(|A_i|-1)\, v'_i(\{g_i\}) \geq |A_j|\, v'_i(\{g_i\}) \geq v'_i(A_j)$ for every agent j, and an agent with an empty bundle envies nobody. A weak Pareto improvement A' on A at $\mathcal{I}'$ gives $c(|A'_i| - |A_i|) \geq 0$ to each agent i with $A_i \neq \emptyset$ by the bound above, and, if $c=-1$, the empty bundle to each other agent; as the bundle sizes sum to m, every agent keeps her bundle size, so the bound gives no agent a strict gain. So $A \in \mathcal{P}(\mathcal{I}')$, and these mr coordinates if $c=1$, and nm if $c=-1$, form a certificate of A. At $\mathcal{J}_0$, the $n-1$ coordinates of the agents $j \neq 1$ with the normal $\mathbf{1}_{M|M}$ certify $(M, \emptyset, \ldots, \emptyset)$: on monotone additive instances they force $v_j = \mathbf{0}$ for every $j \neq 1$, so no agent envies another, no agent other than agent 1 can gain, and agent 1 already holds M. □

Proof of Lemma 5. Let $\mathcal{F} \coloneqq \{S \subseteq M \mid v_2(S) \leq v_2(M \setminus S)\}$, which contains X or $M \setminus X$ for every $X \subseteq M$, let $S^* \in \mathcal{F}$ maximize v_1 over $\mathcal{F}$ and, among the maximizers, minimize v_2, and let $A \coloneqq (S^*, M \setminus S^*)$. For goods, this is the allocation of Aziz [13, Lemma 3]. As $S^* \in \mathcal{F}$, agent 2 does not envy agent 1, and $v_2(A_2) \geq v_2(M)/2$.

For PO, let $C \in \mathcal{A}_0(\mathcal{I})$ satisfy $v_i(C_i) \geq v_i(A_i)$ for each $i \in \{1,2\}$. Then $v_2(C_2) \geq v_2(M)/2 \geq v_2(C_1)$, so $C_1 \in \mathcal{F}$ and $v_1(C_1) \leq v_1(S^*)$; hence $v_1(C_1) = v_1(S^*)$, so $v_2(C_1) \geq v_2(S^*)$, that is, $v_2(C_2) \leq v_2(A_2)$, and no agent is strictly better off in C.

For EFX_0, the conditions of Definition 7 hold for the pairs (i,i), and for agent 2, who does not envy agent 1: $v_2(A_2) \geq v_2(A_1) \geq v_2(A_1 \setminus \{g\})$ for $g \in M^+_{2,1}$, and $v_2(A_2 \setminus \{g\}) > v_2(A_2) \geq v_2(A_1)$ for $g \in M^-_2$. Suppose that agent 1 violates one of them. Then some $g \in M^+_{1,2} \subseteq A_2$ satisfies $v_1(A_2 \setminus \{g\}) > v_1(A_1)$, and we let $X \coloneqq A_1 \cup \{g\}$, or some $g \in M^-_1 \subseteq A_1$ satisfies $v_1(A_1 \setminus \{g\}) < v_1(A_2)$, and we let $X \coloneqq A_1 \setminus \{g\}$. In the first case, $v_1(X) = v_1(A_1) + v_1(\{g\}) > v_1(A_1)$ and $v_1(M \setminus X) = v_1(A_2 \setminus \{g\}) > v_1(A_1)$; in the second, $v_1(X) > v_1(A_1)$, as $v_1(\{g\}) < 0$, and $v_1(M \setminus X) = v_1(A_2) + v_1(\{g\}) > v_1(A_1 \setminus \{g\}) + v_1(\{g\}) = v_1(A_1)$. As X or $M \setminus X$ lies in $\mathcal{F}$, this contradicts the choice of S^*.

For the maximin shares, $v_2(A_2) \geq v_2(M)/2 \geq \mu_{v_2}(2)$, as one part of every $B \in \Pi_2(M)$ is worth at most $v_2(M)/2$. If $B \in \Pi_2(M)$ satisfies $\min_t v_1(B_t) = \mu_{v_1}(2)$, then B_1 or B_2 lies in $\mathcal{F}$, so $v_1(A_1) = v_1(S^*) \geq \mu_{v_1}(2)$. Hence $A \in \mathrm{EFX}_0(\mathcal{I}) \cap \mathrm{PO}(\mathcal{I}) \cap \text{1-out-of-2-MMS}(\mathcal{I})$. □

Proof of Theorem 31. Let $\mathcal{I} = (\{1,2\}, M, (v_1, v_2)) \in \mathcal{C}$ be the unknown instance, and let $\mathcal{F}$ be as in the proof of Lemma 5. Theorem 3, applied to v_2, computes $\mathcal{F}$ with $O\left(m^2 \log m\right)$ comparisons. As a comparison of X with Y for $-v_1$ is one of Y with X for v_1, Theorem 5, applied to $-v_1$, finds $S' \in \arg\max_{S\in\mathcal{F}} v_1(S)$ with $O(m \log m \log(|\mathcal{F}|+1)) = O\left(m^2 \log m\right)$ comparisons, as $|\mathcal{F}| \leq 2^m$. Then Theorem 2, applied to v_1 with the pivot S', gives $\mathcal{T} \coloneqq \{S \in \mathcal{F} \mid v_1(S) = v_1(S')\}$, and Theorem 5,

applied to v_2, finds $S^* \in \arg\min_{S \in \mathcal{T}} v_2(S)$, with $O\left(m^2 \log m\right)$ comparisons each. This S^* is as in the proof of Lemma 5, so $(S^*, M \setminus S^*) \in \mathrm{EFX}_0(\mathcal{I}) \cap \mathrm{PO}(\mathcal{I}) \cap \text{1-out-of-2-MMS}(\mathcal{I})$. □

Algorithm 15 An algorithm to compare absolute values of an additive function.

Input: A finite set M, disjoint subsets $X, Y \subseteq M$, and a queryable additive function $v : 2^M \to \mathbb{R}$.
Output: Whether $|v(X)| < |v(Y)|$ or not.

```
if a comparison query verifies that v(X) < v(Y) then
    if a comparison query verifies that v(∅) < v(X ∪ Y) then
        return True.
else if a comparison query verifies that v(Y) < v(X) then
    if a comparison query verifies that v(X ∪ Y) < v(∅) then
        return True.
return False.
```

Proof of Theorem 32. Suppose that $\mathcal{C}$ consists of homogeneous additive instances, and let $\mathcal{I} \in \mathcal{C}$ be the unknown instance, where $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right)$. We show that Algorithm 2 of Aleksandrov and Walsh [8] can be emulated on $\mathcal{I}$ using $O(|M| \log |M|)$ queries.

First, one computes the ordering of items in Line 2 of Algorithm 2 of Aleksandrov and Walsh [8], i.e., a bijection $\sigma : M \to \{1, 2, \ldots, |M|\}$ such that any pair $(g, h) \in M^2$ satisfies that $\sigma(g) < \sigma(h)$ if $(|v_1(\{g\})|, v_1(\{g\}))$ is lexicographically larger than $(|v_1(\{h\})|, v_1(\{h\}))$. This requires $O(|M| \log |M|)$ queries because Algorithm 15 allows one to compare $|v_1(\{g\})|$ and $|v_1(\{h\})|$ using at most three queries for any distinct $g, h \in M$; if neither absolute value is smaller than the other, then $v_1(\{h\}) = \pm v_1(\{g\})$, and one more query, whether $v_1(\emptyset) < v_1(\{g\})$, tells whether g may precede h.

It remains to check the query complexity for the rest of Algorithm 2 of Aleksandrov and Walsh [8]. By making $O(m)$ queries in advance, Lines 5, 9, and 13 can be run without additional queries. In addition, one can maintain an ordered list on $\{A_i \mid i \in N\}$ sorted by v_i, which costs $O(\log \min\{|N|, |M|\})$ queries at Line 14 but allows Lines 7 and 10 to run without additional queries. Let A'' be the allocation that the emulation returns, which is complete and lies in $\mathrm{EFX}_0(\mathcal{I})$. Let $Z \coloneqq \{g \in M \mid v_1(\{g\}) = v_1(\emptyset)\}$, which $2|M|$ queries find, and let A be obtained from A'' by moving all items of Z into a bundle of A'' of minimum value under v_1, which the ordered list gives. Moving items of value 0 changes neither the value of any bundle under v_1, with or without one item removed, nor the sets $M_{i,j}^+$ and M_i^- of Definition 7, so $A \in \mathrm{EFX}_0(\mathcal{I})$, as every agent has the valuation v_1. Moreover, $A \in \mathrm{EFX}_0^+(\mathcal{I})$: allowing equality in $M_{i,j}^+$ adds the items $g \in A_j \cap Z$, which lie in a bundle A_j of minimum value, so that $v_1(A_i) \geq v_1(A_j) = v_1(A_j \setminus \{g\})$. Finally, every complete allocation is PO at the homogeneous instance $\mathcal{I}$ by the proof of Proposition 4, so $A \in \mathrm{EFX}_0^+(\mathcal{I}) \cap \mathrm{PO}(\mathcal{I}) \subseteq \mathrm{EFX}_0(\mathcal{I}) \cap \mathrm{PO}(\mathcal{I})$, which is found with $O(|M| \log |M|)$ queries in total. By Theorem 19, these bounds are optimal up to a logarithmic factor if $m > n$, as $\mathrm{EFX}_0 \cap \mathrm{PO} = \mathrm{EFX}_0$ on homogeneous instances, and an algorithm that finds an allocation in $\mathrm{EFX}_0^+ \cap \mathrm{PO}$, which is never empty there, also finds one in EFX_0.

For the second claim, let $\mathcal{C}$ consist of homogeneous additive instances, let $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$ be the unknown instance, with $v_i = v$ for every $i \in N$, and let $\mu \coloneqq \mu_v(n)$. We first show that $\left(\mathrm{EFX}_0^+ \cap \mathrm{PO} \cap \text{1-out-of-}n\text{-MMS}\right)(\mathcal{I}) \neq \emptyset$, which is also known in other forms: Chen and Liu [46] show that a leximin allocation is EFX for goods and chores with identical valuations, and Hosseini et al. [94, Corollary 3] that EFX and MMS allocations exist for mixed items with identical additive valuations. Let $Z \coloneqq \{g \in M \mid v(\{g\}) = 0\}$, and let $B \in \Pi_n(M \setminus Z)$ have the lexicographically largest vector of part values sorted in nondecreasing order; its least value is μ, as adding the items

of Z to any part changes no value. Suppose that B violates a condition of Definition 7 for agents i and j, and let $a \coloneqq v(B_i)$ and $b \coloneqq v(B_j)$. Then either some $g \in B_j$ with $v(\{g\}) > 0$ satisfies $v(B_j \setminus \{g\}) > a$, and moving g from B_j to B_i gives the values $a + v(\{g\})$ and $b - v(\{g\})$, or some $g \in B_i$ with $v(\{g\}) < 0$ satisfies $v(B_i \setminus \{g\}) < b$, and moving g from B_i to B_j gives the values $a - v(\{g\})$ and $b + v(\{g\})$. In both cases, both new values exceed a, and $b > a$. So the move keeps the number of parts of value less than a and decreases the number of parts of value a, which makes the sorted vector lexicographically larger, a contradiction. Hence $B \in \mathrm{EFX}_0$ for the items of $M \setminus Z$; after the items of Z are moved into a part of least value, which changes no value, the argument in the first part of this proof shows that B becomes an allocation in $\mathrm{EFX}_0^+(\mathcal{I})$, which lies in 1-out-of-n-MMS$(\mathcal{I})$, as its least value is μ, and in $\mathrm{PO}(\mathcal{I})$.

Next, as in the proof of Theorem 23, Lemma 23 and Theorem 7 find some $S^* \subseteq M$ with $v(S^*) = \mu$ and the signs of $v(T) - v(S^*)$ for all $T \subseteq M$, with $O\left(\min\left\{m^3 \log m, m^2 k \log k \log m\right\}\right)$ comparisons, where $k \coloneqq \min\{n, m\}$. Let D be the set of additive functions x on 2^M for which every $x(T) - x(S^*)$ has the same sign as $v(T) - v(S^*)$, a nonempty convex set, and let $\mathcal{C}'$ be the set of homogeneous additive instances with agents N, items M, and a common valuation in D. For every $x \in D$, $\mu_x(n) = x(S^*)$, as some partition has all its parts worth at least $x(S^*)$ and every partition has a part worth at most $x(S^*)$, which these signs determine; so $\mathcal{F} \coloneqq \{T \subseteq M \mid x(T) \geq x(S^*)\}$ does not depend on $x \in D$, and 1-out-of-n-MMS consists of the complete allocations whose bundles all lie in $\mathcal{F}$ on all of $\mathcal{C}'$. So $\mathcal{P}' \coloneqq \mathrm{EFX}_0^+ \cap \mathrm{PO} \cap$ 1-out-of-n-MMS is nonempty on $\mathcal{C}'$, by the above, and it is verified on $\mathcal{C}'$ with $O(m)$ comparisons, as membership in $\mathcal{F}$ and PO need none. Indeed, given a complete allocation A, find, among its at most $|M| + 1$ distinct bundles, some A_{i_0} of least and some A_{i_1} of largest value, and the signs of all $v(\{g\})$, with $O(m)$ comparisons. Each condition of Definition 7 for a pair (i, j) is strongest for the observer i_0 or for the target i_1, so $A \in \mathrm{EFX}_0^+(\mathcal{I})$ if and only if $v(A_{i_0}) \geq v(A_j \setminus \{g\})$ for all $j \in N$ and $g \in A_j$ with $v(\{g\}) \geq 0$, and $v(A_i \setminus \{g\}) \geq v(A_{i_1})$ for all $i \in N$ and $g \in A_i$ with $v(\{g\}) < 0$, which $|M|$ more comparisons test. So Corollary 1, applied in its homogeneous form to $\mathcal{C}'$, finds an allocation in $\mathcal{P}'(\mathcal{I})$ with $O\left(m \cdot m^2 / \log m\right)$ more comparisons, which gives the bound. □

For *absolutely identical* valuations, those with $|v_i(\{g\})| = |v_j(\{g\})|$ for all agents i and j and items g, Aleksandrov and Walsh [8, Algorithm 2 and Theorem 3] compare the utilities of different agents, but these comparisons can be compiled into comparisons to one agent.

Proposition 6. *Let $\mathcal{C}$ consist of additive instances with absolutely identical valuations. Then $\mathrm{SR}_{\det}(\mathcal{P} \cap \mathrm{PO}) = O(nm + m \log m)$ for each $\mathcal{P} \in \left\{\mathrm{EFX}_0, \mathrm{EFX}_0^+\right\}$. If $\mathcal{C}$ is the set of all such instances with n agents and m items, and only comparison queries are allowed, then $\mathrm{SR}_{\mathrm{rand}}(\mathcal{P} \cap \mathrm{PO}) \geq (n-1)\,m$ and $\mathrm{SR}_{\varepsilon}(\mathcal{P} \cap \mathrm{PO}) \geq ((1-\varepsilon)\,n - 1)\,m$ for every $\varepsilon \in [0, 1/2)$.*

Proof of Proposition 6. Let $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$ be the unknown instance. First, comparing $\{g\}$ with $\emptyset$ in both directions for every $i \in N$ and $g \in M$ gives the sign $\sigma_{i,g} \in \{-1, 0, 1\}$ of each $v_i(\{g\})$ with $2|N||M|$ comparisons. Let w be the additive function with $w(\{g\}) \coloneqq \max_{i \in N} v_i(\{g\})$, whose sign θ_g is $\max_{i \in N} \sigma_{i,g}$; as $|v_i(\{g\})|$ does not depend on i, $w(\{g\}) = \theta_g \sigma_{1,g} v_1(\{g\})$ for every $g \in M$. So, for disjoint $X, Y \subseteq M$, $w(X) - w(Y) = v_1(P) - v_1(Q)$ for the disjoint sets

$$P \coloneqq \{g \in X \mid \theta_g \sigma_{1,g} = 1\} \cup \{g \in Y \mid \theta_g \sigma_{1,g} = -1\}, \tag{34}$$

$$Q \coloneqq \{g \in X \mid \theta_g \sigma_{1,g} = -1\} \cup \{g \in Y \mid \theta_g \sigma_{1,g} = 1\}, \tag{35}$$

and at most two comparisons to agent 1 compare $w(X)$ with $w(Y)$.

Aleksandrov and Walsh [8, Algorithm 2] order the items by nonincreasing $|w(\{g\})|$, placing an item with $w(\{g\}) > 0$ before one with $w(\{h\}) < 0$ of the same absolute value. They then

give each item g in turn to an agent of least current utility among those who value it positively if $w(\{g\}) > 0$, to an agent of largest current utility if $w(\{g\}) < 0$, and to any agent otherwise, as every agent then values it at 0. So every item goes to an agent who values it at $w(\{g\})$, and the current utility of each agent is the value under w of her current bundle; by their Theorem 3, the output A'' lies in $\mathrm{EFX}_0(\mathcal{I}) \cap \mathrm{PO}(\mathcal{I})$. As $|w(\{g\})| = |v_1(\{g\})|$, Algorithm 15 compares absolute values, the known signs θ_g order two items of equal absolute value, and so the ordering takes $O(|M| \log |M|)$ comparisons, and each item then needs $O(|N|)$ comparisons of current utilities, which are comparisons of w between disjoint bundles. Let j minimize $w\left(A''_j\right)$, which $|N|$ more comparisons find, and let A be obtained from A'' by moving every item g with $\theta_g = 0$, which every agent values at 0, into A''_j. As every agent values these items at 0, this changes neither the value to any agent of any bundle, with or without one item removed, nor the sets $M^+_{i,k}$ and M^-_i of Definition 7, so $A \in \mathrm{EFX}_0(\mathcal{I})$; and, as no utility changes, a Pareto improvement on A would be one on A'', so $A \in \mathrm{PO}(\mathcal{I})$. Moreover, $v_i(A_i) = w(A_i) \geq w(A_j) \geq v_i(A_j)$ for every $i \in N$, as $v_i(\{g\}) \leq w(\{g\})$ for every $g \in M$, so no agent envies agent j; allowing equality in $M^+_{i,k}$ adds only items $g \in A_j$, with $v_i(A_j \setminus \{g\}) = v_i(A_j) \leq v_i(A_i)$, so $A \in \mathrm{EFX}^+_0(\mathcal{I})$. In total, $O(nm + m \log m)$ comparisons are made.

For the lower bounds, let $N \coloneqq \{1, 2, \ldots, n\}$, $M \coloneqq \{1, 2, \ldots, m\}$, $\delta \coloneqq 2^{-m-1}$, and $\varepsilon \in [0, 1/2)$. Let $\mathcal{I}_0$ be the instance in which every agent values each $g \in M$ at $-\delta 2^{g-1}$, and, for $(i, g) \in N \times M$, let $\mathcal{I}_{i,g}$ be obtained from $\mathcal{I}_0$ by changing the value of g to agent i to $\delta 2^{g-1}$; these instances lie in $\mathcal{C}$. At $\mathcal{I}_{i,g}$, every PO allocation gives g to agent i, as otherwise moving g from its holder, who values it negatively, to agent i makes both strictly better off, and $(\mathcal{P} \cap \mathrm{PO})(\mathcal{I}_{i,g}) \neq \emptyset$ by the above; so each output is valid for at most m of the nm instances $\mathcal{I}_{i,g}$. By Lemma 14 with $c \coloneqq \mathbf{0}$, $U \coloneqq M$, $\sigma_h \coloneqq -1$ and $r(h) \coloneqq h - 1$ for $h \in M$, the instance $\mathcal{I}_{i,g}$ answers a comparison differently from $\mathcal{I}_0$ only if it is a comparison to agent i whose normal h satisfies $t(h) = g$, so the sets $\Delta(\mathcal{I}_0, \mathcal{I}_{i,g})$ are pairwise disjoint. So Lemma 12 with unit weights and $a \coloneqq m$ gives $\mathrm{SR}_\varepsilon(\mathcal{P} \cap \mathrm{PO}) \geq (1 - \varepsilon)\, nm - m$, and $\varepsilon = 0$ gives $\mathrm{SR}_{\mathrm{rand}}(\mathcal{P} \cap \mathrm{PO}) \geq (n - 1)\, m$. □

Proof of Theorem 33. Suppose that $\mathcal{C}$ consists of monotone additive instances; as noted at the beginning of Appendix B, we may assume that it contains every such instance with at most n agents and at most m items, or every homogeneous one if $\mathcal{C}$ consists of homogeneous instances, so that, for the agents and items of the unknown instance, it is of the form required by Theorem 1, with the nonnegative orthant as every domain. Let $\mathcal{P} \in \{\mathrm{EFL}, \mathrm{EFX}^+_{\mathrm{m}}\}$. Then $(\mathcal{P} \cap \mathrm{RMMS}_m)(\mathcal{I}) \neq \emptyset$ for every $\mathcal{I} \in \mathcal{C}$, by Feige [65, Observation 7 and Corollary 8] and Akrami et al. [6, Proposition 1.1], whose EFX for partial allocations requires envy to vanish after the removal of any item, which is EFX^+_m on monotone instances, and whose EFL is that of Definition 6; running the verifiers of Proposition 2 and Theorem 22 one after the other gives $\mathrm{VR}_{\mathrm{det}}(\mathcal{P} \cap \mathrm{RMMS}_m) = O\left(nm + nm^2 \log m\right) = O\left(nm^2 \log m\right)$. So Corollary 1 gives $\mathrm{SR}_{\mathrm{det}}(\mathcal{P} \cap \mathrm{RMMS}_m) = O\left(nm^2 \log m \cdot nm^2 / \log(nm)\right)$, as $x / \log x$ increases for $x \geq e$.

Next, let $\mathcal{C}$ consist of monotone additive instances with two agents, and let $\mathcal{I} \in \mathcal{C}$ be the unknown instance, with agents 1 and 2, valuations v_1 and v_2, and items M. Theorem 7 with $d \coloneqq 2$ finds $B \in \Pi_2(M)$ with $\min_t v_1(B_t) = \mu_{v_1}(2)$ with $O\left(m^2 \log m\right)$ comparisons; one more comparison orders its parts so that $v_1(B_1) \leq v_1(B_2)$, and moving into B_1 every item g with $v_1(\{g\}) = 0$, which $|M|$ comparisons find, changes no value under v_1. Agent 2 then takes a part that she values at least as much as the other, with one more comparison, and agent 1 receives the other part; let A be the resulting complete allocation. Agent 2 does not envy agent 1, so, as v_2 is monotone, she satisfies the conditions of EFX^+_0 in Definition 7, and $v_2(A_2) \geq v_2(M)/2 \geq \mu_{v_2}(2)$. Agent 1 receives a part worth at least $\mu_{v_1}(2)$, and does not envy agent 2 if she receives B_2. If she receives

B_1 and some $g \in B_2$ satisfies $v_1(B_2 \setminus \{g\}) > v_1(B_1)$, then $v_1(\{g\}) > 0$, as $g \notin B_1$, so both parts of $(B_1 \cup \{g\}, B_2 \setminus \{g\})$ are worth more than $\mu_{v_1}(2)$ to agent 1, a contradiction; so she satisfies the conditions of EFX_0^+ as well. Hence $A \in \mathrm{EFX}_0^+(\mathcal{I}) \cap \text{1-out-of-2-MMS}(\mathcal{I})$, which is contained in $\mathrm{EFX}_m^+(\mathcal{I}) \cap \mathrm{RMMS}_m(\mathcal{I})$ and in $\mathrm{EFL}(\mathcal{I}) \cap \mathrm{RMMS}(\mathcal{I})$, as $\mathrm{EFX}_0^+(\mathcal{I}) \subseteq \mathrm{EFL}(\mathcal{I})$ for monotone $\mathcal{I}$ (Definition 7) and $\rho_{v_i}(2) = \mu_{v_i}(2)$ for each i (Lemma 21).

Now let $\mathcal{C}$ consist of homogeneous monotone additive instances, let $\mathcal{I} = \left(N, M, (v_i)_{i \in N}\right) \in \mathcal{C}$ be the unknown instance, with $v_i = v$ for every $i \in N$, and let $\mu \coloneqq \mu_v(|N|)$. As in the proof of Theorem 23, Lemma 23 and Theorem 7 find some $B \in \Pi_{|N|}(M)$ with $\min_t v(B_t) = \mu$, together with $(\mathrm{sign}(v(T) - \mu))_{T \subseteq M}$, with $O\left(\min\left\{m^3 \log m, m^2 \min\{n, m\} \log \min\{n, m\} \log m\right\}\right)$ comparisons. The allocations below give every agent a bundle worth at least $\mu \geq \rho_v(|N|)$ (Lemma 21), and they need $O(m \log m)$ more comparisons.

For EFX_m^+, give each agent t an inclusion-minimal $C_t \subseteq B_t$ with $v(C_t) \geq \mu$, which these signs determine. Then $v(C_j \setminus \{g\}) < \mu \leq v(C_i)$ for all $i, j \in N$ and $g \in C_j$, so, as v is monotone, $C \in \mathrm{EFX}_m^+(\mathcal{I}) \cap \mathrm{RMMS}_m(\mathcal{I})$.

For EFL, call $g \in M$ large if $v(\{g\}) > \mu$, which these signs determine. Fewer than $|N|$ items are large, as otherwise $|N|$ of them in distinct parts, with the other items added to any parts, would give a partition into $|N|$ parts that are all worth more than μ. While some part of B contains two large items, some other part contains none; replace that part with a singleton of one of these large items, and release its former items. Then, from every part with a large item, release the other items, and, from every other part, release all items outside an inclusion-minimal subset worth at least μ. Every part remains worth at least μ throughout, no released item is large, and the parts that remain form a partial allocation C in which $v(C_j \setminus \{g\}) \leq \mu$ for every $j \in N$ and $g \in C_j$. Finally, give the released items one by one to an agent whose current bundle has least value, which a heap of the at most $|M| + 1$ distinct bundles does with $O(m \log m)$ comparisons, and let A be the resulting complete allocation, in which every agent receives a bundle worth at least μ. Let $i, j \in N$ with $|A_j| > 1$. If $A_j = C_j$, then every $g \in A_j$ satisfies $v(A_j \setminus \{g\}) \leq \mu \leq v(A_i)$ and $v(\{g\}) \leq \mu$, as C_j is not a large singleton. Otherwise, let g be the last released item that agent j receives; when she receives it, $A_j \setminus \{g\}$ has least value among the bundles, so $v(A_i) \geq v(A_j \setminus \{g\})$, and $v(A_i) \geq \mu \geq v(\{g\})$, as g is not large. So g witnesses the condition of Definition 6 for (i, j), and $A \in \mathrm{EFL}(\mathcal{I}) \cap \text{1-out-of-}n\text{-MMS}(\mathcal{I}) \subseteq \mathrm{EFL}(\mathcal{I}) \cap \mathrm{RMMS}(\mathcal{I})$, as A is complete. □

B.6 Section 3.6

In the following proofs, let $N \coloneqq \{1, 2, \ldots, n\}$ and $M \coloneqq \{1, 2, \ldots, m\}$, and let $\varepsilon \in [0, 1/2)$. For a finite set S, an integer $t \geq 1$, and a map $d : \binom{S}{t} \to [0, 1/2]$, the *layer function* $h_{t,d} : 2^S \to \mathbb{R}$ given by $h_{t,d}(Y) \coloneqq \min\{|Y|, t\}$ if $|Y| \neq t$ and $h_{t,d}(Y) \coloneqq t - d(Y)$ if $|Y| = t$ is monotone and submodular: adding an item to a bundle of fewer than $t-1$ items, of $t-1$ items, of t items, and of more than t items raises its value by 1, by $1 - d \in [1/2, 1]$, by $d \in [0, 1/2]$, and by 0, respectively, so marginal values are nonnegative and do not increase along inclusion, which characterizes monotone submodular functions. For a monotone submodular function f on 2^S with $S \subseteq M$, the function $X \mapsto f(X \cap S)$ on 2^M, its sum with a monotone additive function, and its minimum $\min\{c, f\}$ with a constant c are monotone and submodular; for the last, the marginal value of an item g at $X \subseteq X'$ with $g \notin X'$ satisfies $\min\{c, f(X \cup \{g\})\} - \min\{c, f(X)\} \geq \min\{c, f(X' \cup \{g\})\} - \min\{c, f(X')\}$, as the right-hand side is 0 if $f(X') \geq c$, and otherwise $f(X) \leq f(X') < c$, so that the inequality follows from $c - f(X) \geq c - f(X')$ and the one for f. If alternatives to a baseline instance each raise the values of a set of pairs of an agent and a bundle, and these sets are pairwise disjoint, then their sets Δ are pairwise disjoint, for comparison and value queries alike (Appendix A.6); the same holds if each alternative lowers its values instead.

Proof of Theorem 34. Let $\mathcal{P}$ be as in the statement. As a search algorithm also decides, it suffices to bound $\mathrm{DC}_\varepsilon(\mathcal{P})$, and $\varepsilon = 0$ gives the bounds on $\mathrm{DC}_{\mathrm{rand}}(\mathcal{P})$. Let $t \coloneqq \lfloor k/2 \rfloor \geq 1$, $S \coloneqq \{1, 2, \ldots, k\}$, and $G \coloneqq M \setminus S$, so that $|G| = n - 2$, and call the items of G *large*. Let $\mathcal{Y} \coloneqq \left\{ Y \in \binom{S}{t} \,\middle|\, 1 \notin Y \right\}$ if k is even and $\mathcal{Y} \coloneqq \binom{S}{t}$ if k is odd, so that $|\mathcal{Y}| = L$, as $\binom{k-1}{t} = \binom{k}{t}/2$ if $k = 2t$; moreover, $L \geq 2^{k-1}/(k+1)$, as $\binom{k}{\lfloor k/2 \rfloor}$ is the largest of the $k+1$ binomial coefficients $\binom{k}{j}$, which sum to 2^k. For $Z \in \mathcal{Y} \cup \{\emptyset\}$, let $d_Z(Y) \coloneqq 1/2$ if $Y \in \mathcal{Y} \setminus \{Z\}$ and $d_Z(Y) \coloneqq 0$ otherwise, for $Y \in \binom{S}{t}$, and let $\mathcal{I}_Z$ be the homogeneous instance with agents N, items M, and the common valuation

$$v_Z(X) \coloneqq \min\{t, h_{t,d_Z}(X \cap S) + t|X \cap G|\} \qquad \forall X \subseteq M, \tag{36}$$

which is monotone and submodular by the above. Every bundle has value at most t, and a bundle X has value t if and only if it contains a large item, $|X| \geq t+1$, or $X \in \binom{S}{t} \setminus (\mathcal{Y} \setminus \{Z\})$; otherwise, its value is at most $t - 1/2$. As $\mathcal{I}_Z$ is homogeneous, a complete allocation is EF at $\mathcal{I}_Z$ if and only if all its bundles have the same value.

First, $\mathrm{EF}(\mathcal{I}_\emptyset) = \emptyset$. As $|G| = n - 2$, every complete allocation has at least two bundles without a large item. If all bundles had value t at $\mathcal{I}_\emptyset$, each of these two would have at least $t+1$ items of S, or, if k is even, exactly t items of S including item 1; as at most one of them contains item 1, they would have at least $2t + 1 > k$ items of S if k is even and at least $2t + 2 > k$ if k is odd. So the bundles of an EF allocation would have a common value below t; then none of them contains a large item, so that $n = 2$ and $k = m$, and both have at most t items, as larger bundles have value t, so that $k = 2t$ and both have exactly t items; but then the one that contains item 1 has value t.

Next, for $Y \in \mathcal{Y}$, let A^Y be the complete allocation that gives one large item to each of agents $1, 2, \ldots, n-2$, Y to agent $n-1$, and $S \setminus Y$ to agent n. At $\mathcal{I}_Y$, it gives every agent the largest value t, as $S \setminus Y$ is a member of $\binom{S}{t}$ that contains item 1 if k is even and has $t+1$ items if k is odd, so it is EF and PO there. Hence $\mathcal{P}(\mathcal{I}_Y) \supseteq (\mathrm{EF} \cap \mathrm{PO})(\mathcal{I}_Y) \neq \emptyset$, whereas $\mathcal{P}(\mathcal{I}_\emptyset) \subseteq \mathrm{EF}(\mathcal{I}_\emptyset) = \emptyset$, so the inputs $\mathcal{I}_\emptyset$ and $\mathcal{I}_Y$ of decision have different correct answers. Moreover, v_Y differs from $v_\emptyset$ only at the bundle Y, whose value it raises from $t - 1/2$ to t, so the sets $\Delta(\mathcal{I}_\emptyset, \mathcal{I}_Y)$ for $Y \in \mathcal{Y}$ are pairwise disjoint, and Lemma 12 with unit weights gives $\mathrm{DC}_\varepsilon(\mathcal{P}) \geq (1 - 2\varepsilon) L$.

For the term n, let $U \coloneqq \{1, 2, \ldots, n\} \subseteq M$, let $\mathcal{J}$ be the homogeneous instance with agents N, items M, and the common valuation $u(X) \coloneqq \min\{|X \cap U|, 1\}$, and, for each $g \in U$, let $\mathcal{J}_g$ be the one with the common valuation u_g given by $u_g(X) \coloneqq 1/2$ if $X \cap U = \{g\}$ and $u_g(X) \coloneqq u(X)$ otherwise. These valuations are the layer functions on U with $t = 1$, and $d \equiv 0$ or d that is $1/2$ at $\{g\}$ and 0 elsewhere, applied to $X \cap U$, so they are monotone and submodular. At $\mathcal{J}$, giving item i to each agent $i \in N$ and the items of $M \setminus U$ to agent 1 gives every agent the largest value 1, so $\mathcal{P}(\mathcal{J}) \supseteq (\mathrm{EF} \cap \mathrm{PO})(\mathcal{J}) \neq \emptyset$. At $\mathcal{J}_g$, the bundles of a complete allocation cannot all have value 0, as some bundle contains an item of U, nor all have value $1/2$, as only the bundle that contains g can, nor all have value 1, as then each bundle would contain an item of U and the one that contains g another one, which needs $n+1$ items of U; so $\mathcal{P}(\mathcal{J}_g) \subseteq \mathrm{EF}(\mathcal{J}_g) = \emptyset$. As u_g lowers u exactly at the bundles X with $X \cap U = \{g\}$, which differ for distinct g, the sets $\Delta(\mathcal{J}, \mathcal{J}_g)$ for $g \in U$ are pairwise disjoint, and Lemma 12 with unit weights gives $\mathrm{DC}_\varepsilon(\mathcal{P}) \geq (1 - 2\varepsilon) n$.

For the term $\binom{n+1}{2}$, let $m > n$ and $U' \coloneqq \{1, 2, \ldots, n+1\} \subseteq M$, and let $\mathcal{J}'$ be the homogeneous instance with agents N, items M, and the common valuation $w(X) \coloneqq \min\{|X \cap U'|, 2\}$, the layer function on U' with $t = 2$ and $d \equiv 0$, applied to $X \cap U'$. For each $B \in \binom{U'}{2}$, let $\mathcal{J}'_B$ be the one with the common valuation w_B given by $w_B(X) \coloneqq 1$ if $X \cap U' = B$ and $w_B(X) \coloneqq w(X)$ otherwise. Then $w_B(X) = \min\{c_B(X \cap U'), 2\}$, where $c_B(Y)$ is the number of the sets B and $\{g\}$ for $g \in U' \setminus B$ that meet Y; as c_B is a sum of the monotone submodular functions $Y \mapsto \min\{|Y \cap C|, 1\}$ for these sets C, it is monotone and submodular, and so is w_B, by the above. At $\mathcal{J}'$, some bundle of a

complete allocation contains an item of U', so the bundles of an EF allocation would all have value 1 or all have value 2, and would thus contain exactly one item of U' each, or at least two each; as $n \neq n+1$ and $2n > n+1$, $\mathrm{EF}(\mathcal{J}') = \emptyset$. At $\mathcal{J}'_B$, giving B and the items of $M \setminus U'$ to agent 1, and one item of $U' \setminus B$ to each other agent, gives every agent the value 1, so $\mathrm{EF}(\mathcal{J}'_B) \neq \emptyset$. As w_B lowers w exactly at the bundles X with $X \cap U' = B$, which differ for distinct B, the $\binom{n+1}{2}$ sets $\Delta(\mathcal{J}', \mathcal{J}'_B)$ are pairwise disjoint, and Lemma 12 with unit weights gives $\mathrm{DC}_\varepsilon(\mathrm{EF}) \geq (1-2\varepsilon)\binom{n+1}{2}$. This term does not extend to $\mathrm{EF} \cap \mathrm{PO}$: at $\mathcal{J}'_B$, every EF allocation gives the bundles value 1, as $2n > n+1$, so one bundle X has $X \cap U' = B$, and moving an item of B from it to another bundle raises the value of that bundle to 2 and keeps the value of X. □

Proof of Theorem 35. The proofs of Theorems 25 and 26 use only monotone additive instances, which are monotone submodular, so they give $\mathrm{SR}_{\mathrm{rand}}(\mathrm{PO}) \geq n-1$, $\mathrm{VR}_{\mathrm{rand}}(\mathrm{PO}) \geq n-1$, $\mathrm{SR}_\varepsilon(\mathrm{PO}) \geq (1-\varepsilon)n-1$, and $\mathrm{VR}_\varepsilon(\mathrm{PO}) \geq (1-2\varepsilon)(n-1)$ here, even if value queries are allowed. For the other terms, let $t \coloneqq \lfloor m/2 \rfloor \geq 1$, so that $K = \binom{m}{t}$. For $\mathcal{F} \subseteq \binom{M}{t}$, let $d_{\mathcal{F}}(Y) \coloneqq 0$ if $Y \in \mathcal{F}$ and $d_{\mathcal{F}}(Y) \coloneqq 1/2$ otherwise, and let $\mathcal{I}_{\mathcal{F}}$ be the instance with agents N and items M in which agent 1 has the valuation $h_{t,d_{\mathcal{F}}}$, agent 2 has the valuation $X \mapsto \min\{|X|, m-t\}$, which is the layer function $h_{m-t,0}$, and every other agent has the valuation $\mathbf{0}$; these valuations are monotone and submodular. Let $\mathcal{I}^\circ \coloneqq \mathcal{I}_\emptyset$ and $\mathcal{I}^Y \coloneqq \mathcal{I}_{\{Y\}}$ for $Y \in \binom{M}{t}$. Agents 1 and 2 value every bundle at most at t and $m-t$, respectively.

At $\mathcal{I}^Y$, the allocation $A^Y \coloneqq (Y, M \setminus Y, \emptyset, \ldots, \emptyset)$ gives every agent her largest value. In a complete allocation that gives agent 2 the value $m-t$, she receives at least $m-t$ items, so agent 1 receives at most t items, which she values at t only if they form Y, and then agent 2 receives $M \setminus Y$. So A^Y is the only complete allocation that gives every agent her largest value, and it is a Pareto improvement on every other one; hence $\mathrm{PO}(\mathcal{I}^Y) = \{A^Y\}$, and each output is valid for at most one of the K inputs $\mathcal{I}^Y$. As $\mathcal{I}^Y$ raises only the value of the pair $(1, Y)$ above $\mathcal{I}^\circ$, the sets $\Delta(\mathcal{I}^\circ, \mathcal{I}^Y)$ are pairwise disjoint, and Lemma 12 with unit weights and $a = 1$ gives $\mathrm{SR}_\varepsilon(\mathrm{PO}) \geq (1-\varepsilon)K - 1$, and $\mathrm{SR}_{\mathrm{rand}}(\mathrm{PO}) \geq K - 1$ with $\varepsilon = 0$. For $\mathcal{P}$ as in the statement, $\mathcal{P}(\mathcal{I}^Y) = \{A^Y\}$, as $\mathcal{P}(\mathcal{I}^Y) \subseteq \mathrm{PO}(\mathcal{I}^Y) = \{A^Y\}$ and A^Y gives every agent her largest value, so the same argument gives these bounds for $\mathcal{P}$.

For verification, let $s \coloneqq \lceil m/2 \rceil$, so that $\binom{M}{s}$ has K members, let $\mathcal{J}$ be the homogeneous instance with agents N, items M, and the common valuation $u \coloneqq h_{s,d}$ with $d \equiv 1/2$, and, for $Y \in \binom{M}{s}$, let $\mathcal{J}_Y$ be the one with the common valuation $u_Y \coloneqq h_{s,d_Y}$, where d_Y is 0 at Y and $1/2$ elsewhere; these instances are in $\mathcal{C}$, and they are homogeneous. At $\mathcal{J}$, bundles of fewer than s items have integer values below s, those of exactly s items have value $s - 1/2$, and the others have value s. Let A° be a complete allocation that gives $s+1 \leq m$ items to agent 1, the other $m-s-1 < s$ items to agent 2, and nothing to the other agents, so that $u(A^\circ_1) = s$ and $u(A^\circ_2) = m-s-1$. At $\mathcal{J}$, a bundle has value at least s only if it has at least $s+1$ items, and value at least $m-s-1$ only if it has at least $m-s-1$ items, so, in a complete allocation in which agents 1 and 2 are not worse off than in A°, they receive exactly $s+1$ and $m-s-1$ items, with the same values as in A°, and the other agents receive nothing; hence $A^\circ \in \mathrm{PO}(\mathcal{J})$. For $Y \in \binom{M}{s}$, neither bundle of A° has s items, so the agents value their bundles of A° at $\mathcal{J}_Y$ as at $\mathcal{J}$, and the complete allocation that gives Y to agent 1 and $M \setminus Y$ to agent 2 gives agent 1 the value s and agent 2 the value $s - 1/2$ if $m = 2s$ and $s-1$ if $m = 2s-1$, both above $m-s-1$; so it is a Pareto improvement on A°, and $A^\circ \notin \mathrm{PO}(\mathcal{J}_Y)$. As u_Y raises u only at the bundle Y, the K sets $\Delta((\mathcal{J}, A^\circ), (\mathcal{J}_Y, A^\circ))$ are pairwise disjoint, so Lemma 12 with unit weights gives $\mathrm{VR}_\varepsilon(\mathrm{PO}) \geq (1-2\varepsilon)K$, and $\mathrm{VR}_{\mathrm{rand}}(\mathrm{PO}) \geq K$ with $\varepsilon = 0$, even on homogeneous instances. □

Proof of Remark 12. If $m \leq n$, let A give each item to a different agent at a homogeneous monotone

submodular instance with valuation v. By submodularity, $v(X) \leq v(\emptyset) + \sum_{g \in X} (v(\{g\}) - v(\emptyset))$ for every bundle X, with equality if $|X| \leq 1$, so the values of the bundles of every complete allocation sum to at most $\sum_{g \in M} v(\{g\}) + (n-m)\, v(\emptyset)$, with equality at A; as a Pareto improvement on A would have a larger sum, A is PO.

Let k, L, t, S, G, $\mathcal{Y}$, $\mathcal{I}_Z$, v_Z, and A^Y be as in the proof of Theorem 34, and let $\mathcal{P} \in \{\text{PO}, 1\text{-out-of-}n\text{-MMS}\}$. For $Y \in \mathcal{Y}$, as A^Y gives every agent the largest value t at $\mathcal{I}_Y$, it is a Pareto improvement on every complete allocation that does not, and $\mu_{v_Y}(n) = t$; so $\mathcal{P}(\mathcal{I}_Y)$ is the set $\mathcal{T}_Y$ of complete allocations whose bundles all have value t at $\mathcal{I}_Y$. In an allocation of $\mathcal{T}_Y$, at least two bundles contain no large item, and each of them has at least $t+1$ items, or is Y, or, if k is even, is a member of $\binom{S}{t}$ that contains item 1; as at most one of them contains item 1, and $2t+1 > k$ if k is even and $2t+2 > k$ if k is odd, one of them is Y. As $v_{Y'}(Y) = t - 1/2$ for $Y' \in \mathcal{Y} \setminus \{Y\}$, the sets $\mathcal{T}_Y$ for $Y \in \mathcal{Y}$ are pairwise disjoint, and each output is valid for at most one of the L inputs $\mathcal{I}_Y$. With the pairwise disjoint sets $\Delta(\mathcal{I}_\emptyset, \mathcal{I}_Y)$ of that proof, Lemma 12 with unit weights and $a = 1$ gives $\text{SR}_\varepsilon(\mathcal{P}) \geq (1-\varepsilon)\, L - 1$, and $\text{SR}_{\text{rand}}(\mathcal{P}) \geq L - 1$ with $\varepsilon = 0$.

For the verification of 1-out-of-n-MMS, fix $Z \in \mathcal{Y}$, and let $A^\circ \coloneqq A^Z$. At $\mathcal{I}_\emptyset$, it gives agent $n-1$ the value $t - 1/2$ and every other agent the value t. As $\text{EF}(\mathcal{I}_\emptyset) = \emptyset$, every complete allocation has a bundle of value below t, hence at most $t - 1/2$, so $\mu_{v_\emptyset}(n) \leq t - 1/2$ and $A^\circ \in 1\text{-out-of-}n\text{-MMS}(\mathcal{I}_\emptyset)$. For $Y \in \mathcal{Y} \setminus \{Z\}$, the agents value their bundles of A° at $\mathcal{I}_Y$ as at $\mathcal{I}_\emptyset$, as $S \setminus Z \notin \mathcal{Y}$ if k is even, so agent $n-1$ has value $t - 1/2 < t = \mu_{v_Y}(n)$, and $A^\circ \notin 1\text{-out-of-}n\text{-MMS}(\mathcal{I}_Y)$. The $L-1$ sets $\Delta((\mathcal{I}_\emptyset, A^\circ), (\mathcal{I}_Y, A^\circ))$ are pairwise disjoint, so Lemma 12 with unit weights gives $\text{VR}_\varepsilon(1\text{-out-of-}n\text{-MMS}) \geq (1-2\varepsilon)(L-1)$, and $\text{VR}_{\text{rand}}(1\text{-out-of-}n\text{-MMS}) \geq L - 1$ with $\varepsilon = 0$. □

Proof of Proposition 5. A function $f : 2^M \to \{0,1\}$ is *monotone* if $f(X) \leq f(X')$ whenever $X \subseteq X'$. By Hansel's theorem [84] [see 12, Theorem 1], some algorithm determines every monotone f with at most $\binom{m}{\lfloor m/2 \rfloor} + \binom{m}{\lfloor m/2 \rfloor + 1} \leq 2K$ queries of values $f(X)$. For a monotone valuation v_i and a bundle R, the functions $X \mapsto [v_i(X) \geq v_i(R)]$ and $X \mapsto [v_i(X) > v_i(R)]$ are monotone, and each of their values is the answer to the comparison of whether $v_i(R) \leq v_i(X)$, or the negation of the answer to that of whether $v_i(X) \leq v_i(R)$; so each of them can be determined with at most $2K$ comparisons, which we call *learning* it.

For verification, an allocation that is not complete is not PO and is rejected without queries. For a complete allocation A, learning both functions for $R = A_i$ for every agent i, with at most $4nK$ comparisons, determines, for every complete allocation A', whether it is a Pareto improvement on A, and thus whether A is PO.

For search, we choose bundles $R_1, R_2, \ldots, R_n$ in turn. For $j \in N$, given $R_1, \ldots, R_{j-1}$ and the learned functions $X \mapsto [v_i(X) \geq v_i(R_i)]$ for $i < j$, let $\mathcal{B}_j$ be the set of complete allocations A' with $v_i(A'_i) \geq v_i(R_i)$ for every $i < j$, which is known, and let $\mathcal{F}_j \coloneqq \left\{ A'_j \,\middle|\, A' \in \mathcal{B}_j \right\}$. If $\mathcal{B}_j \neq \emptyset$, then $\mathcal{F}_j$ is nonempty and closed under taking subsets, as moving items from agent j to another agent keeps an allocation in $\mathcal{B}_j$, by monotonicity. As v_j is monotone, some inclusion-maximal member R_j of $\mathcal{F}_j$ maximizes v_j over $\mathcal{F}_j$; as these maximal members form an antichain, there are at most K of them by Sperner's theorem, so R_j is found with at most $K-1$ comparisons, and with none if $j = 1$, as $\mathcal{F}_1 = 2^M$. If $j < n$, learn $X \mapsto [v_j(X) \geq v_j(R_j)]$; then $\mathcal{B}_{j+1} \neq \emptyset$, as it contains an allocation of $\mathcal{B}_j$ that gives R_j to agent j. Finally, output some $A \in \mathcal{B}_n$ with $A_n = R_n$, using at most $(n-1)(2K + K - 1) \leq 3(n-1)K$ comparisons in total, as agent 1 needs none to find R_1 and agent n learns no function. For every $j \in N$, as $A \in \mathcal{B}_n \subseteq \mathcal{B}_j$, we have $A_j \in \mathcal{F}_j$, so $v_j(A_j) \leq v_j(R_j)$, and also $v_j(A_j) \geq v_j(R_j)$, as $A \in \mathcal{B}_{j+1}$ if $j < n$ and $A_n = R_n$; so $v_j(A_j) = v_j(R_j) = \max_{X \in \mathcal{F}_j} v_j(X)$. If A' were a Pareto improvement on A, let j be the least agent with $v_j\left(A'_j\right) > v_j(A_j)$; then $v_i(A'_i) \geq v_i(A_i) = v_i(R_i)$ for $i < j$, so $A' \in \mathcal{B}_j$ and

$v_j\left(A'_j\right) \leq v_j\left(R_j\right) = v_j\left(A_j\right)$, a contradiction; so A is PO.

Now let the valuations also be submodular. Then $v_i\left(X\right) \leq v_i\left(\emptyset\right) + \sum_{g \in X}\left(v_i\left(\{g\}\right) - v_i\left(\emptyset\right)\right)$ for every $i \in N$ and bundle X, so $v_i\left(X\right) > v_i\left(\emptyset\right)$ if and only if $v_i\left(\{g\}\right) > v_i\left(\emptyset\right)$ for some $g \in X$. In search, at stage j, the union U_j of the members of $\mathcal{F}_j$ is known, and its singletons are members of $\mathcal{F}_j$, so one comparison tells whether $\max_{X \in \mathcal{F}_j} v_j\left(X\right) > v_j\left(\emptyset\right)$, which holds if and only if $v_j\left(U_j\right) > v_j\left(\emptyset\right)$, as, v_j being submodular, $v_j\left(U_j\right) > v_j\left(\emptyset\right)$ gives some $g \in U_j$ with $v_j\left(\{g\}\right) > v_j\left(\emptyset\right)$. If not, let $R_j \coloneqq \emptyset$, which maximizes v_j over $\mathcal{F}_j$ and needs no learned function, as $v_j\left(X\right) \geq v_j\left(\emptyset\right)$ for every bundle X; otherwise, proceed as above. The argument above still applies, and every agent j with $R_j \neq \emptyset$ receives a nonempty bundle in the output, as $v_j\left(A_j\right) = v_j\left(R_j\right) > v_j\left(\emptyset\right)$; so at most $\min\left\{n, m\right\}$ agents j have $R_j \neq \emptyset$, and the comparisons number at most $n + 3\min\left\{n, m\right\}K$. In verification, let J be the set of agents i with $A_i \neq \emptyset$, so that $\left|J\right| \leq \min\left\{n, m\right\}$, and learn both functions only for $i \in J$, with at most $4\min\left\{n, m\right\}K$ comparisons. These determine the set $\mathcal{B}$ of complete allocations A' with $v_i\left(A'_i\right) \geq v_i\left(A_i\right)$ for every $i \in J$, in which, by monotonicity, no agent is worse off than in A, and whether some $A' \in \mathcal{B}$ gives an agent of J more than A does, in which case A is not PO. Otherwise, A is PO unless some $A' \in \mathcal{B}$ gives an agent $j \notin J$ more than $v_j\left(\emptyset\right)$. The set $\mathcal{F} \coloneqq \left\{A'_j \,\middle|\, A' \in \mathcal{B}\right\}$ is the same for every $j \in N \setminus J$, as swapping the bundles of two agents outside J keeps an allocation in $\mathcal{B}$, and it is closed under taking subsets, as moving items to an agent of $J \neq \emptyset$ does; so, with the union U of its members, such an A' exists for j if and only if $v_j\left(U\right) > v_j\left(\emptyset\right)$, which one comparison tells for each $j \in N \setminus J$, for at most $n + 4\min\left\{n, m\right\}K$ comparisons in total.

Finally, let $\mathcal{C}$ be the set of all homogeneous monotone submodular instances with n agents and $m < 2n$ items, and let v be the common valuation, so that a complete allocation is EF if and only if all its bundles have the same value; it suffices to find all EF allocations. If $m < n$, every complete allocation has an empty bundle, so, by monotonicity and submodularity, an EF allocation exists if and only if $v\left(M\right) \leq v\left(\emptyset\right)$, in which case v is constant and every complete allocation is EF; one comparison decides this. If $m = n$, an EF allocation with an empty bundle gives every bundle the value $v\left(\emptyset\right)$, so every singleton has this value by monotonicity, and giving each agent one item is EF as well; hence an EF allocation exists if and only if all singletons have the same value, which $2m$ comparisons decide. Otherwise, $s \coloneqq m - n$ satisfies $1 \leq s < n$. Find an item h with the largest value $\alpha \coloneqq v\left(\{h\}\right)$ among the singletons, find the items g with $v\left(\{g\}\right) = \alpha$, which we call *high*, and compare α with $v\left(\emptyset\right)$, with $2m - 1$ comparisons. If $\alpha = v\left(\emptyset\right)$, then v is constant by the inequality above and monotonicity, and every complete allocation is EF. Otherwise, the bundles of an EF allocation are nonempty, as the one that contains h has value at least $\alpha > v\left(\emptyset\right)$, so, as $m < 2n$, one of them is a singleton, and their common value is α. Let ℓ be the number of items that are not high, which we call *low*; in an EF allocation, a bundle X with $r \geq 1$ low items is not a singleton, so $\left|X\right| - 1 \geq r/2$, and, as the numbers $\left|X\right| - 1$ over its bundles X sum to s, we have $\ell \leq 2s$, and every bundle of j low items and no high item satisfies $j - 1 + \left(\ell - j\right)/2 \leq s$, i.e., $j \leq 2s + 2 - \ell$. So there is no EF allocation if $\ell > 2s$; otherwise, compare $v\left(\{g, g'\}\right)$ with $v\left(\{g\}\right)$ for every high item g and item $g' \neq g$, and $v\left(X\right)$ with α in both directions for every set X of $j \in \{2, 3, \ldots, 2s + 2 - \ell\}$ low items. As $\alpha \leq v\left(X\right) \leq \alpha + \sum_{g' \in X \setminus \{g\}}\left(v\left(\{g, g'\}\right) - \alpha\right)$ for every bundle X that contains a high item g, by monotonicity and submodularity, these comparisons determine whether $v\left(X\right) = \alpha$ for every bundle X that can be in an EF allocation, and thus all EF allocations. They number $O\left(m^2\right)$ plus at most twice $\sum_{j=0}^{2s+2-\ell}\binom{\ell}{j} \leq \sum_{p=0}^{2s+2}\sum_{j \geq 0}\binom{p-j}{j} = \sum_{p=0}^{2s+2} F_{p+1} < F_{2s+5} \leq \varphi^{2s+4}$, where F_r is the r-th Fibonacci number, with $F_1 = F_2 = 1$; the first inequality maps each j to $p \coloneqq \ell + j$. $\square$

C Additional Related Work

Tables 2 to 4 list our bounds for search, decision, and verification, together with existing ones.

C.1 Fair and Efficient Allocation of Indivisible Items

A central thread in discrete fair division studies the compatibility of approximate envy-freeness with economic efficiency. For goods with additive valuations, Caragiannis et al. [42] showed that a maximum Nash welfare allocation is both EF1 and PO, which establishes the existence of fair and efficient allocations in one of the most fundamental settings.

As maximizing Nash welfare is computationally hard, Barman et al. [26] developed techniques based on integral Fisher markets and price adjustments: a pseudopolynomial-time algorithm for EF1 + PO, a non-constructive proof that EF1 + fPO allocations exist, and a polynomial-time 1.45-approximation of the maximum Nash welfare. Subsequent work refined this market-based viewpoint and broadened the range of simultaneous guarantees: Barman and Krishnamurthy [23] studied the proximity of markets with integral equilibria; Aziz et al. [17] gave a polynomial-time algorithm for PO and almost proportional allocations; Freeman et al. [72] considered approximate equitability together with PO; and Garg and Murhekar [75] computed EF1 + fPO and EQ1 + fPO allocations of goods in pseudopolynomial time, and faster for instances with few distinct values, while Garg and Murhekar [74] gave a polynomial-time algorithm for EFX + fPO allocations of two-valued instances, although Liu and Zhang [111] show that the algorithm of its conference version may fail to terminate, and give a polynomial-time algorithm for weighted EFX + fPO.

The chores setting requires different techniques, as the Nash-welfare approach and many market arguments for goods do not transfer directly. A series of works obtained EF1 + PO or related guarantees under structured disutilities, including bivalued (easy and difficult) chores [76, 60], two types of chores [19], three agents or two disutility functions [77], and weighted EF1 with few types of agents or chores [78].

Mahara [119] gave a polynomial-time algorithm for EF1 + fPO allocations of goods with a fixed number of agents, and Mahara [120] recently proved that EF1 + PO, and even EF1 + fPO, allocations of additive chores always exist. Together with the result of Caragiannis et al. [42], this establishes the existence of EF1 + PO allocations for both monotone and antitone additive instances. However, these works assume full cardinal access to the valuations or use non-constructive arguments, and they do not address comparison-query complexity.

Mixed manna, where the same item may be a good for some agents and a chore for others, presents additional obstacles. Aziz et al. [18] studied the fair allocation of indivisible goods and chores in a unified framework, and showed several limits of extending techniques for goods or chores alone. More recent work therefore introduces fairness notions that remain compatible with efficiency for mixed manna. For example, Barman and Verma [24] introduced introspective EF1 (IEF1) and its extrospective counterpart XEF1, and proved that IEF1 + fPO and XEF1 + fPO allocations exist for additive mixed manna; XEF1 and IEF1 coincide with EF1 on monotone and antitone instances, respectively (Definition 8). For EF1 itself, fPO is out of reach: Mackenzie and Suzuki [117, Theorem 4.3] give an additive mixed-manna instance with three agents and four items, robust to small perturbations, in which no allocation is both EF1 and fPO, although one is EF1 and PO; whether EF1 + PO allocations always exist for additive mixed manna remains open [24, Section 1.1].

Our work builds on this compatibility but asks an orthogonal, information-theoretic question: instead of assuming that the valuations are given in full, we ask whether fair and efficient allocations can be found with comparisons of bundles only.

Table 2: Bounds on the number of comparison queries needed to find an allocation with a given property, or to report that none exists, including existing ones; Table 1 selects from these. The number of items is denoted by m, and $K \coloneqq \binom{m}{\lfloor m/2 \rfloor}$. Abbreviations: Hom. = homogeneous; Abs. Id. = absolutely identical; Mon. = monotone; Ant. = antitone; Pos. = with positive item values; Add. = additive; Submod. = submodular; det. = deterministic.

Property	Agents	Valuation	Upper Bound	Lower Bound
EF	2	Add.	$O\left(m^2 \log m\right)$ [Thm. 8]	$\Omega\left(m^2\right)$ [Thm. 12]
	2 (Hom.)	Mon. Submod.	–	$\binom{m}{\lfloor m/2 \rfloor}/2$ [Thm. 34], cf. [127]
	$2n$ (Hom.)	Add.	$O\left(m^2 \log m\right)$ [Thm. 8]	$\Omega\left(n + (m-2n)^2\right)$, $m \geq 2n$ [Thm. 13]
	n (Hom.)	Add.	$O\left(m^{2.5} \log^2 m\right)$ [Thm. 8]	$\Omega\left(n + (m-n)^2\right)$, $m \geq n$ [Thm. 13]
	n (Hom.)	Mon./Ant. Add.	$\tilde{O}\left(m + (m-n)^2\right)$, $n \leq m < 2n$ [Thm. 9]	
	n	Add., $v_i(M) > 0$	$n\, 2^{m-n} m^{O(1)}$, $m \geq n$ [Thm. 9]	$2^{\Omega(m-n)}$, $n \geq 3$ [Thm. 14]
	n	Add.	$n\, 2^{O(m)}$ [Cor. 4]	$\Omega\left(n \min\{n, m\}\right)$ [Thm. 10]
	n	Mon. Add.	$n + 2^{O(m)}$ [Cor. 4]	$\Omega\left(\min\{n^2, m^2\} + n\right)$ [Thm. 11]
				$2^{\Omega(m)}$, $n \geq 3$ [Thm. 14]
	n (Hom.)	Mon. Submod.	$O\left(m^2 + 2.62^{m-n}\right)$, $m < 2n$ [Prop. 5]	$\Omega\left(n^2 + 2^{m-n}/m\right)$, $m > n$ [Thm. 34]
EFX_0	n (Hom.)	Add.	$O\left(m \log m\right)$ [Thm. 32]	$\Omega\left(m\right)$, $m > n$ [Thm. 19]
	n	Mon. Add.	–	$\Omega\left(n\,(m-n)\right)$ [Thm. 18]
EFX_0^+	2	Add.	$O\left(m \log m\right)$ [Thm. 16]	$\Omega\left(m\right)$ [Thm. 19]
EFX_{n-3}^+	3, 4	Mon. Add.	$O\left(m^3/\log m\right)$ [Thm. 16]	–
EFX_{n-2}^+	n	Mon. Add.	$O\left(n^2 m^3/\log(nm)\right)$ [Thm. 16]	–
EF1	2	Any	$\lceil \log_2(m-1) \rceil + 3$ [Thm. 15]	
	3	Mon. Add.	$O\left(\log m\right)$ [127, 39]	$\Omega\left(\log m\right)$ (det.) [127, 39]
	3	Mon.	$O\left(\log^2 m\right)$ (value queries) [127]	
	3	Ant. Add.	$O\left(\log m\right)$ [109]	–
	n	Mon. Add.	$O\left(nm \log(m/n)\right)$, $m \geq 2n$ [107]	$\Omega\left(n \log(m/n)\right)$ (det.) [127]
				$\Omega\left(n \min\{n, m-n\}\right)$ [Thm. 17]
	n	Doubly Mon./Add.	$O\left(nm \log \min\{n, m\}\right)$ [Thm. 15]	$\Omega\left(n \min\{n, m-n\}\right)$ [Thm. 17]
	n (Hom.)	Mon. Add.	$O\left(n^2 \log m\right)$ [39]	$\Omega\left(\log(m/n)\right)$ (det.) [cf. 127]
	n (Hom.)	Mon.	$O\left(n \log n \log(m/n)\right)$, $m > 2n$ [158]	
EFL	n	Mon.	$O\left(nm \log \min\{n, m\}\right)$ [Thm. 15]	$\Omega\left(n \min\{n, m-n\}\right)$ [Thm. 17]
1-out-of-n MMS	2	Add.	$O\left(m^2 \log m\right)$ [Thm. 21]	$\Omega\left(m^2\right)$ [Cor. 5]
	n (Hom.)	Add.	$\tilde{O}\left(m^2 \min\{n, m\}\right)$ [Thm. 21]	$\Omega\left((m-n)^2\right)$, $m > n$ [Cor. 5]
	n (Hom.)	Mon. Submod.	–	$\Omega\left(2^{m-n}/m\right)$, $m > n$ [Rem. 12]
1-out-of-d MMS	n	Add.	$O\left(nm^3 \log m\right)$ [Thm. 21]	–
	n	Mon. Add.	$\tilde{O}\left(n\left(m + (m-d)^3\right)\right)$, $m/2 < d \leq m$ [Thm. 21]	–
RMMS	2	Add.	$O\left(m^2 \log m\right)$ [Thm. 23]	$\Omega\left(m^2\right)$ [Cor. 5]
	n (Hom.)	Add.	$\tilde{O}\left(m^2 \min\{n, m\}\right)$ [Thm. 23]	
	n	Add.	$O\left(nm^3 \log m\right)$ [Thm. 23]	$\Omega\left((m-n)^2\right)$, $m > n$ [Cor. 5]
	n	Mon. Add.	$\tilde{O}\left(n\left(m + (m-n)^3\right)\right)$, $m/2 < n < m$ [Thm. 23]	
PO	n	Mon. Add.	$O\left(n + m \log n\right)$ [Prop. 4]	$\Omega\left(n + m \log n\right)$ [Thm. 25]
	n	Add.	$O\left(nm\right)$ [Prop. 4]	$\Omega\left(nm\right)$ [Thm. 28]
	n	Ant. Add.	$O\left(nm\right)$ [Prop. 4]	
	n (Hom.)	Add.	0 [Prop. 4]	–
	n	Mon.	$O\left(nK\right)$ [Prop. 5]	$\Omega\left(n + K\right)$ [Thm. 35]
	n	Mon. Submod.	$O\left(n + \min\{n, m\}\, K\right)$ [Prop. 5]	
	n (Hom.)	Mon. Submod.	$O\left(n + \min\{n, m\}\, K\right)$ [Prop. 5]	$\Omega\left(2^{m-n}/m\right)$, $m > n$ [Rem. 12]
EF + PO	n	Mon. Submod.	–	$\Omega\left(K\right)$ [Thm. 35]
	n (Hom.)	Mon. Submod.	–	$\Omega\left(n + 2^{m-n}/m\right)$, $m \geq n$ [Thm. 34]
XEF1/IEF1 + PO	2	Add.	$O\left(m^2 \log m\right)$ [Thm. 31]	$\Omega\left(m\right)$ [Thm. 30]
	n	Add.	$O\left(n \min\{n, m\}\, m^4\right)$ [Thm. 29]	$\Omega\left(nm\right)$ [Thm. 30]
EF1 + PO	n	Mon. Add.	$O\left(n \min\{n, m\}\, m^4\right)$ [Thm. 29]	$\Omega\left(n + m \min\{n, m\}\right)$ [Thm. 30]
	$n \geq m$	Ant. Add.	$O\left(nm\right)$ [Rem. 10]	$\Omega\left(nm\right)$ [Thm. 30]
EFX_0 + PO + MMS	2	Add.	$O\left(m^2 \log m\right)$ [Thm. 31]	$\Omega\left(m^2\right)$ [Cor. 5]
EFX_0^+ + PO	n (Hom.)	Add.	$O\left(m \log m\right)$ [Thm. 32]	$\Omega\left(m\right)$, $m > n$ [Thm. 19]
	n (Abs. Id.)	Add.	$O\left(nm + m \log m\right)$ [Prop. 6]	$\Omega\left(nm\right)$ [Prop. 6]
EFX_0^+ + PO + MMS	n (Hom.)	Add.	$O\left(m^3/\log m + m^2 \min\{n, m\} \log^2 m\right)$ [Thm. 32]	$\Omega\left((m-n)^2\right)$, $m > n$ [Cor. 5]
$\text{EFL}/\text{EFX}_m^+$ + RMMS_m	2	Mon. Add.	$O\left(m^2 \log m\right)$ [Thm. 33]	$\Omega\left(m^2\right)$ [Cor. 5]
	n (Hom.)	Mon. Add.	$\tilde{O}\left(m^2 \min\{n, m\}\right)$ [Thm. 33]	$\Omega\left((m-n)^2\right)$, $m > n$ [Cor. 5]
	n	Mon. Add.	$O\left(n^2 m^4\right)$ [Thm. 33]	
Any verifiable	n	Add., $\leq V$ values	$O\left(n\,(m+V) \log(V+1)\right)$ [Cor. 3]	–

Table 3: Bounds on the number of comparison queries needed to decide whether an allocation with a given property exists, including existing ones. Properties that every instance in the class satisfies, such as EF1 on monotone instances and PO, need no queries and are omitted. The number of items is denoted by m. Abbreviations are as in Table 2.

Property	Agents	Valuation	Upper Bound	Lower Bound
EF	2	Add.	$O\left(m^2 \log m\right)$ [Thm. 8]	$\Omega\left(m^2\right)$ [Thm. 12]
	2 (Hom.)	Mon. Submod.	–	$\binom{m}{\lfloor m/2 \rfloor}/2$ [Thm. 34], cf. [127]
	n (Hom.)	Add.	$O\left(m^{2.5} \log^2 m\right)$ [Thm. 8]	$\Omega\left(n + (m-n)^2\right)$, $m \geq n$ [Thm. 13]
	n (Hom.)	Mon./Ant. Add.	$\tilde{O}\left(m + (m-n)^2\right)$, $n \leq m < 2n$ [Thm. 9]	
	$n = m$ (Hom.)	Pos. Add.	n; $O\left(\log\left(1/\varepsilon\right)\right)$, error ε [Rem. 6]	n, zero error [Thm. 13]
	n	Add., $v_i(M) > 0$	$n\, 2^{m-n} m^{O(1)}$, $m \geq n$ [Thm. 9]	$2^{\Omega(m-n)}$, $n \geq 3$ [Thm. 14]
	n	Add.	$n\, 2^{O(m)}$ [Cor. 4]	$\Omega\left(n\right)$; $2^{\Omega(m)}$ if $n \geq 3$ [Thms. 11, 14]
	n	Mon. Add.	$n + 2^{O(m)}$ [Cor. 4]	
	n (Hom.)	Mon. Submod.	$O\left(m^2 + 2.62^{m-n}\right)$, $m < 2n$ [Prop. 5]	$\Omega\left(n^2 + 2^{m-n}/m\right)$, $m > n$ [Thm. 34]
EF + PO	n (Hom.)	Mon. Submod.	–	$\Omega\left(n + 2^{m-n}/m\right)$, $m \geq n$ [Thm. 34]
1-out-of-d MMS	n	Add.	$O\left(nm^3 \log m\right)$ [Thm. 21]	–
Any verifiable	n	Add., $\leq V$ values	$O\left(n\left(m+V\right)\log\left(V+1\right)\right)$ [Cor. 3]	–

C.2 Preference Elicitation and Valuation Learning

Queries that reduce the burden of preference elicitation have been studied extensively in combinatorial auctions and learning theory. Blum et al. [31] relate preference elicitation to query learning, and emphasize that the goal in allocation problems is often not to learn each valuation in full, but to ask enough questions to identify an optimal allocation. Conen and Sandholm [52] proposed incremental elicitation with value, order, and rank queries that narrows down the welfare-maximizing or Pareto-efficient allocations, and Hudson and Sandholm [95] showed, experimentally and theoretically, that incremental elicitors with suitable query types and policies need only a vanishing fraction of the information that full revelation collects as the number of items grows.

Conitzer [53] elicited single-peaked preferences with comparison queries, and recommender systems elicit relative preferences of new users with static questionnaires [141] and choose which pairs of items to ask about by active learning [100]. The works above are close to ours in motivation, but their queries are mostly value, demand, or other auction-specific queries, or comparisons of single alternatives, and their goal is welfare maximization or a ranking, rather than fairness and efficiency under comparisons of bundles.

A related line in learning theory studies how set functions or utility functions can be learned, also from comparisons. Balcan et al. [20] study the learnability of valuation classes such as subadditive and XOS functions up to multiplicative factors, and Balcan et al. [21] learn combinatorial functions from pairwise comparisons, including coverage, submodular, XOS, and subadditive functions. Ge et al. [80] study linear utilities under pairwise comparison queries and distinguish predicting future comparisons from recovering the underlying parameters; active comparisons can recover the parameters where passive data cannot. These works ask when comparisons suffice to learn a comparator or an approximate valuation; our algorithms do not reconstruct the valuations, and they make only the comparisons needed to certify an allocation (Section 2.2).

C.3 Ranking, Sorting, and Selection from Comparisons

Pairwise comparisons have also been studied in ranking and noisy sorting. Braverman and Mossel [37] study noisy sorting without resampling, where each pair is compared only once, and give a polynomial-time algorithm that finds the maximum-likelihood order, which they show to be close to the true order. Shah et al. [142] give minimax rates for estimating latent scores from pairwise

Table 4: Bounds on the number of comparison queries needed to verify whether a given allocation has a given property, including existing ones. The number of items is denoted by m, and $K \coloneqq \binom{m}{\lfloor m/2 \rfloor}$. Abbreviations are as in Table 2.

Property	Agents	Valuation	Upper Bound	Lower Bound
EF	n	Add.	$n \min\{n-1, m\}$ [Prop. 1]	$\Omega(n \min\{n, m\})$ [Thm. 10]
	n	Mon. Add.	$O\left(\min\{n, m\}^2 + n\right)$ [Prop. 1]	$\Omega\left(\min\{n^2, m^2\} + n\right)$ [Thm. 11]
EF1 XEF1	n	Mon. Add.	$O(nm)$ [Props. 2, 3]	$\Omega\left(n \min\{n, m\} \log\left(2 + \frac{m}{n}\right)\right)$ [Thm. 17]
EF1	n	Add.	$O(nm)$ [Prop. 2]	$\Omega(m)$ [Thm. 17]
IEF1	n	Any	$O(nm)$ [Prop. 3]	$\Omega(n \min\{n, m-n\})$ [Thm. 17]
EFL	n	Mon.	$O(nm)$ [Prop. 2]	$\Omega(n \min\{n, m-n\})$ [Thm. 17]
EFX_0	n	Mon. Add.	$O(nm)$ [Prop. 2]	$\Omega(nm)$ [Thm. 18]
	n (Hom.)	Mon. Add.	$2m + n - 1$ [Thm. 19]	$m - 1$, $m > n$ [Thm. 19]
	n	Ant. Add.	$O\left(m + n \min\{n, (m-n)_+\}\right)$ [Prop. 2]	$\Omega\left(m + n \min\{n, (m-n)_+\}\right)$ [Thm. 18]
EFX_0^+	n	Mon. Add.	$O(m \min\{n, m\} + n)$ [Prop. 2]	$\Omega(m \min\{n, m\} + n)$ [Thm. 18]
EFX_c, EFX_c^+	n	Any	$O(nm)$ [Prop. 2]	–
1-out-of-d MMS	n	Add.	$O\left(nm^2 \log m\right)$ [Thm. 20]	–
	n (Hom.)	Add.	$O\left(m^2 \log m\right)$ [Thm. 20]	–
	n (Hom.)	Mon. Submod.	–	$\Omega(2^{m-n}/m)$, $d = n < m$ [Rem. 12]
RMMS_c	n	Add.	$O\left(nm^2 \log m\right)$ [Thm. 22]	–
	n (Hom.)	Add.	$O\left(m^2 \log m\right)$ [Thm. 22]	–
PO	n	Add.	$O\left(nm^2 \log m\right)$ [Thm. 24]	$\Omega\left(nm + m^2\right)$ [Thms. 27, 28]
	n	Ant. Add.		
	n	Mon. Add.	$\tilde{O}\left(n + m^2 \min\{n, m\}\right)$ [Rem. 13]	$\Omega\left(n + m^2\right)$ [Thms. 26, 27]
	n (Hom.)	Add.	0 [Rem. 12]	–
	n	Mon.	$O(nK)$ [Prop. 5]	$\Omega(n + K)$ [Thm. 35]
	n	Mon. Submod.	$O(n + \min\{n, m\} K)$ [Prop. 5]	
	n (Hom.)	Mon. Submod.	$O(n + \min\{n, m\} K)$ [Prop. 5]	$\Omega(K)$ [Thm. 35]
EF1/XEF1/IEF1 + PO	n	Mon. Add.	$\tilde{O}\left(nm + m^2 \min\{n, m\}\right)$ [Rem. 13; Props. 2, 3]	$\Omega(m \min\{n, m\})$ [Thm. 30]
Any verifiable	n	Add., $\leq V$ values	$O(n(m+V)\log(V+1))$ [Cor. 3]	–

comparisons under Bradley–Terry–Luce and Thurstone models, which depend on the spectrum of the Laplacian of the comparison graph.

Ailon [2] studies active ranking from pairwise preferences that may be non-transitive, with the goal of minimizing disagreement with the preference tournament, and Jamieson and Nowak [97] rank items embedded in a Euclidean space by their distance to a reference point, with a number of adaptive comparisons that grows with the dimension and logarithmically with the number of items. These works share our concern with reducing the number of comparisons, but their alternatives form a fixed finite set of items or candidates, whereas we compare exponentially many bundles.

Several papers study selection and ranking under stochastic, imprecise, or adversarial comparisons. Mohajer et al. [121] analyze active top-K ranking from noisy comparisons; Falahatgar et al. [63] give tight bounds for maximum selection and near-tight bounds for ranking under stochastic transitivity assumptions; Ren et al. [135] study exact ranking from noisy pairwise and listwise comparisons, and Ren et al. [136] the selection of the best k items; and Davidson et al. [56] study top-k selection and clustering with erroneous comparisons, motivated by crowdsourcing.

Other models weaken the reliability of comparisons: Ajtai et al. [3] study imprecise comparisons motivated by just-noticeable differences, Acharya et al. [1] maximum selection and sorting with adversarial comparators, Trevisan [150] randomized sorting and selection with adversarial comparators and their trade-offs with the number of rounds, and Vitercik et al. [153] sorting from crowdsourced comparisons supplemented by costly expert verifications. These models are useful points of comparison for noisy or unreliable feedback, but they recover rankings or top sets, whereas we compute allocations that satisfy combinatorial fairness and efficiency constraints.

C.4 Comparison Oracles in Active Learning, Geometry, and Optimization

Comparison queries have recently played an important role in active learning and in geometric decision trees. Kane et al. [101] study active classification with label and comparison queries, and introduce the inference dimension to show when comparisons give exponential savings; Hopkins et al. [91] extend this framework to reliable and noise-tolerant active classification.

Kane et al. [102] study generalized comparison trees for point location in hyperplane arrangements, whose depth is polynomial in the dimension and logarithmic in the number of hyperplanes, and Hopkins et al. [92] make this depth nearly optimal. This geometric perspective is close to our one-pivot point location (Section 2.4), which compares one bundle with each of the exponentially many bundles using polynomially many comparisons.

Other comparison oracles arise in decision making, optimization, and crowdsourcing. Shao et al. [143] learn an unknown preference vector over objectives, for personalized multi-objective decision making, from comparisons of policies or of representative sets of trajectories; Kazemi et al. [105] consider weak comparison oracles that answer only when they are confident, for comparison-based content search; and Cohen et al. [50] study dueling bandits in which teams, rather than single arms, are compared. Tang et al. [147] study zeroth-order optimization with ranking oracles, motivated by learning from human feedback, Tao et al. [148] gradient testing and estimation from comparisons of function values, and Zeng and Shen [160] show that pairwise comparisons can greatly reduce the label complexity of PAC learning from the crowd.

Cohen-Addad et al. [51] introduced exact combinatorial optimization from comparisons: they minimize an unknown linear function over any family of sets with nearly quadratically many comparisons in the size of the ground set, possibly in exponential time, and optimize from certificates of optimality (Sections 2.2 to 2.4 use their results). Their model also covers graph cuts: they find a minimum cut of an unweighted graph in randomized polynomial time with a number of comparisons nearly linear in the number of vertices, adapting the algorithm of Rubinstein et al. [139], which needs only that many queries of cut values, whereas learning the graph needs quadratically many; for weighted graphs, their general result needs only polynomially many comparisons, but possibly exponential time. Fox and Woodruff [70] close this gap: with a number of comparisons cubic in the size of the ground set, they find in polynomial time a nontrivial minimizer of a symmetric submodular function, and thus a weighted minimum cut; as in our model, their comparisons detect ties. Following Cohen-Addad et al. [51], Barat et al. [22] recover the objective of a packing linear program, up to scale and when its values lie on a grid, from comparisons of its optimal solutions under varying constraints, and show that fractional knapsack is as general as packing in this model.

These works show the breadth of comparison-based feedback models, but they typically learn, rank, or optimize a single objective, whereas we handle several unknown set functions coupled by fairness and efficiency predicates.

D AI Methodology

Codex (OpenAI) and Claude Code (Anthropic) assisted with checking and simplifying proofs; proposing concrete proof steps, lower-bound constructions, and sharper bounds; searching the literature; and revising the manuscript. The tools materially affected the write-up of Section 3 and Appendices A.2, A.5, A.6 and B. The author verified the correctness and originality of all content.

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